- Surfaces, $\pi_1(S)$
- G Lie group (reductive: $GL(n,\mathbb{C})$; $SL(n,\mathbb{C})$, $SL(n,\mathbb{R})$ etc...)

Flexibility of surfaces group

① $Rep(\pi_1(S), G) = Hom(\pi_1(S), G)/G$ is non empty

② {smooth points} carries a symplectic structure

(Atiyah-Bott, Goldman)

Other structures? complex structures?

A: (Riemann) If (S,J) is a Riemann surface

$$Rep(\pi_1(S), \mathbb{R}) = \{\text{abelian differentials}, f(z)dz\} = H^0(\underset{T^{*}S}{\underset{\|}{K}})$$

B: (Poincaré, Fricke-Klein)

$$\{\text{complex structures}\} \underset{\text{connected component}}{\subset} Rep(\pi_1(S), PSL(2,\mathbb{R}))$$

Comparison of A & B:

- Both provides link: complex analysis $\leftrightsquigarrow$ topology
- a "superficial difference": $\mathbb{R} \neq SL(2,\mathbb{R})$
- a crucial difference: the second theorem is purely topological.

Goal: move from A to B...

① Hitchin components (and new results)

② Non abelian Hodge theory & harmonic mappings

③ Minimal surfaces in symmetric spaces

I Hitchin components

① to simplify take $G = SL(n,\mathbb{R})$

Fuchsian rep in $SL(n,\mathbb{R})$: $\rho : \pi_1(S) \xrightarrow[\text{faith.}]{\text{discrete}} PSL(2,\mathbb{R}) \xrightarrow{\text{irreducible}} PSL(n,\mathbb{R})$

Hitchin rep in $SL(n,\mathbb{R})$ = deformation of fuchsian.

$\rightsquigarrow \mathcal{H}(S,G)$ = Hitchin component = {Hitchin rep}/conjuga

(Hitchin) $\mathcal{H}(S,G) \approx \mathbb{R}^{-\chi(S)\dim G}$ (true for all simple $\mathbb{R}$-split group)

More about that later

- a Hitchin representation is a quasi-isometry [Anosov interpretation]
- $\mathrm{Mod}(S)$ acts properly on $\mathcal{H}(S,G)$
- smooth symplectic manifold: ω = Atiyah–Bott–Goldman symplectic form.
- (presumably) if $G \neq SL(2,\mathbb{R})$

$$\int_{\mathcal{H}(S,G)/\mathrm{Mod}(S)} \omega^{\wedge \max} = \infty$$

true for , $G = SL(3,\mathbb{R})$ using a version of McShane's identity and Fock–Goncharov identity in the spirit of Mirzakhani.

② What about complex geometry?

Big restriction: G_0 simple non compact $\mathbb{R}$-split Lie group of rank 2

$G_0 = SL(3,\mathbb{R})$, $Sp(4,\mathbb{R})$, $G_{2,2}$ only 3 groups ($SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$)

There is a $\mathrm{Mod}(S)$ equivariant diffeomorphism

$$\mathcal{H}(S,G) = \{(J,q) \mid J \text{ complex structure}, q \in H^0(K^{d(G)})\}$$

$d(G)=3$, $SL(3,\mathbb{R})$; 4 for $Sp(4,\mathbb{R})$; 6 for $G_{2,2}$

[Loftin, L for $SL(3,\mathbb{R})$]

Given a Hitchin representation ρ, $\exists$ a unique homotopy equivalent minimal surface in $\mathrm{Sym}(G)/\rho(\pi_1(S))$.

$\rightarrow$ 2 corollaries

$\mathcal{H}(S,G)$ has a Kähler geometry (rk by Zhang–Kim)

If $\mathrm{MinArea}(\rho) = \mathrm{Inf}\{\mathrm{Area}(S) \mid S \hookrightarrow \mathrm{Sym}(G)/\rho(\pi_1(S))\}$

Then $\mathrm{MinArea}(\rho) \geq -2\pi\chi(S)$, with equality if and only if ρ is Fuchsian

Question is

$$\int_{\mathcal{X}(S,g)/\mathrm{Mod}(S)} e^{-\frac{1}{\hbar}\mathrm{MinArea}(\rho)}\, d\rho < \infty \quad ?$$

when $\hbar \to 0$; one obtains $\approx \hbar^N$ (characteristic number $\mathcal{M}_{g,n}$).

II Non abelian Hodge theory

(Hitchin parametrization $G = SL(n,\mathbb{R})$)

Let (S,J) be a Riemann surface,

$$\mathcal{X}(S,G) \approx \bigoplus_{i=2}^{n} H^0(K^i)$$
$$\hookrightarrow \{ f(z)\, dz^i \}$$

Explain: why do holomorphic differentials appear? = harmonic mappings!

- $f : (S,J) \longmapsto (M,g)$

  the energy $E(f) = \frac{1}{2}\int \|Tf\|_2\, dv_h$. ($h$ conformal to J)

- A harmonic mapping is a critical pt of the energy

- Harmonic mappings satisfy a nice equation (more later)

Assume G is a complex Lie group.

① From representations to harmonic mappings

$\rho : \pi_1(M) \longrightarrow G$; $\exists f : \tilde{M} \to \mathrm{Sym}(G)$

with $f\circ\gamma = \rho(\gamma)\circ f$.

Are there better maps

Corlette theorem: If $\rho(\pi_1(M))$ is reductive then there exists a unique ρ equivariant harmonic mapping.

example $M = S^1$; closed geodesics, avoid cusps.

② From harmonic mappings to Higgs bundles

⇝ satisfy some (Euler–Lagrange) equations: for surfaces this have a nice interpretation

$$f : S \to N$$

$$Tf \in \Omega^1(S, f^*TN)$$

let $E = (f^*TN) \otimes \mathbb{C}$; E is a holomorphic bundle.

let $\omega = (Tf)^{1,0} \in \Omega^{1,0}(S, E)$; (1,0) decoration = complexification

$\boxed{\bar\partial \omega = 0}$ in other words, $\omega \in H^0(K \otimes E) = \{\text{holom. sect. of } K \otimes E\}$, $K = T^*S$

Harmonic mapping into $\mathrm{Sym}(G) = X$, for $G_{\mathbb{C}}$ complex lie simple group

- $(TX) \otimes \mathbb{C}$ is a $\mathfrak{g}_{\mathbb{C}}$ bundle ($\mathfrak{g}_{\mathbb{C}}$ = lie alg.)

ex: for $G = SL(n, \mathbb{C})$, $TX = \{U \in M_n(\mathbb{C}) \mid \mathrm{Tr}(U) = 0 ;\ U = {}^t\bar{U}\}$.

We have associated to a harmonic mapping f into $\mathrm{Sym}(G)$

$$\left.\begin{array}{l} \rightsquigarrow (E, \omega) \\ \quad \omega \hookrightarrow H^0(K \otimes E) \\ \quad E \hookrightarrow \text{holomorphic } \mathfrak{g}_{\mathbb{C}}\text{-vector bundle} \end{array}\right\} =: \text{Higgs bundle.}$$

③ From Higgs bundle to holomorphic differentials

$$(E, \omega) \longmapsto \text{holomorphic differentials}$$

In local coordinates

$$\omega = A(z)\, dz \ , \text{where } \bar\partial A = 0$$

$$\mathrm{tr}(\omega \cdot \cdots \cdot \omega) = \mathrm{tr}(A^q(z))\, dz^q \in H^0(K^q)$$

Even though A is only defined up to conjugacy $\mathrm{tr}(A^q dz^q)$ is well defined.

The Hitchin grand scheme

$$
\begin{tikzcd}
& \{(f,\rho)=\rho \text{ equivariant harmonic mapping}: f:\tilde{S}\to Sym(G_{\mathbb{C}})\} \arrow[dl] \arrow[dr] & \\
\{\text{reductive } \rho:\pi_1(S)\to G_{\mathbb{C}}\} \arrow[ur, bend left, Rightarrow, "\text{Corlette theorem}"] \arrow[d, phantom, "\cap"] & & \{\text{Higgs bundles } (E,\omega)\} \arrow[ul, bend right, Rightarrow, "\text{Hitchin - Simpson theorem}"'] \arrow[dd, shift left, "\text{Hitchin fibration}"] \\
Rep(\pi_1(S,G_{\mathbb{C}})) \arrow[d, phantom, "\cup"] & & \\
Rep(\pi_1(S),G_{\mathbb{R}}) & & \text{Hitchin base} = \bigoplus_{q=2}^{n} H^0(K^q) \arrow[ll, "\text{Hitchin parametrisation theorem}"'] \arrow[uu, shift left, "\text{Hitchin section}"]
\end{tikzcd}
$$

Rks :

- the black arrows are "local constructions"
- the blue double arrows are "global" constructions requiring that S is closed and involving solving elliptic PDE's on S. They are homomorphisms.
- Observe the Hitchin base has ½ the dimension of $Rep(\Gamma, G_{\mathbb{C}})$
- The orange arrow is a composition other arrows. It is a homeomorphism onto $Rep(\Gamma, G_{\mathbb{R}})$ where $G_{\mathbb{R}}$ is the real split form of $G_{\mathbb{C}}$: that is $G_{\mathbb{R}}$ is the least compact of all groups G such that $\mathfrak{g}\otimes\mathbb{C} = \mathfrak{g}_{\mathbb{C}}$; it is also the

unique such group which admits a Cartan subalgebra whose elements are $\mathbb{R}$-split ($=\mathrm{Ad}(x)$ is diagonalisable over $\mathbb{R}$).

III Minimal surfaces

① General situation.

[L.] . If ρ is a quasi-isometry, there exists a ρ-equivariant minimal surface.

Uniqueness in general is not true ($G = PSL(2,\mathbb{C})$)

(Zheng Huang — Biao Wang for the quasi-Fuchsian case)

② Hitchin components

$$\rho \rightsquigarrow \underset{\text{har. mapping}}{f} \text{ equivariant} \rightsquigarrow (q_2, \dots, q_n) \text{ holomorphic differentials.}$$

Fact : $q_2(f) = 0 \Leftrightarrow f$ is a minimal mapping

Let $\mathcal{E} \to T(S)$ be the vector bundle over Teichmüller space whose fiber at a complex structure J is

$$\mathcal{E}_J = \bigoplus_{q \geq 2}^{n} H^0(K^q)$$

by the previous remark $\mathcal{E} = \{\text{equivariant minimal surfaces}\}$

Using Hitchin parametrisation for all J, we get a natural map

$$\mathcal{E} \longmapsto \mathrm{Rep}(\pi_1(S), G)$$

Question : is this a bijection.

[L.] If G real split rk 2, uniqueness holds

Idea of the proof

$$\begin{tikzcd} & G/T \arrow[d] \\ S \arrow[ur] \arrow[r] & G/K \end{tikzcd}$$

minimal surfaces equivariant w.r to Hitchin lift to (nice) holomorphic curves in G/T

Key difficulty : prove the stability of minimal surfaces

much easier to prove stability of holomorphic curve!

builds up on work by Baraglia, (cousin to some work of Bolton-Pedit

Woodward)