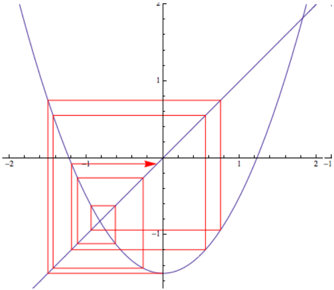
On W. Thurston's core-entropy theory

Bill in Jackfest, Feb. 2011 "*It started out... in 1975 or so. My first job after graduate school was, I was the assistant of Milnor...*

At one point, I'd gotten a programmable HP desktop calculator and a pen plotter, I started fooling around with logistic maps..."

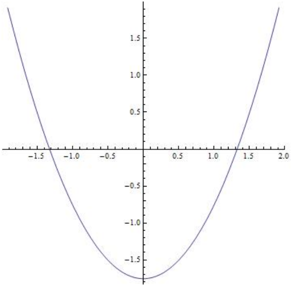


presented by Tan Lei (Université d'Angers, France) Cornell, June 2014

Classic Milnor-Thurston kneading theory

Let $f : I \to I$ continuous, by Misiurewicz-Szlenk,

$$h_{top}(f, I) := \lim_{n \to \infty} \frac{\log\{\#\text{laps of } f^n\}}{n}$$



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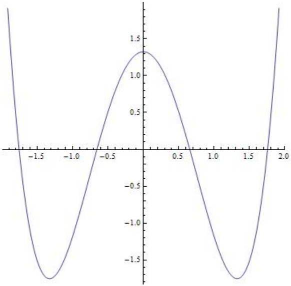

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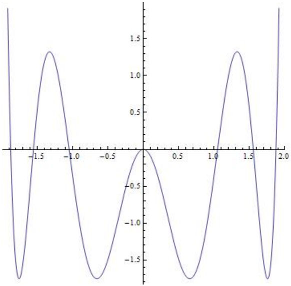

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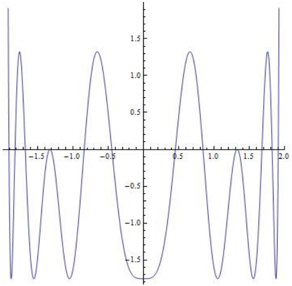

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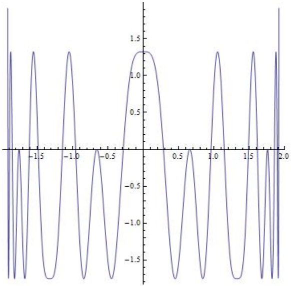

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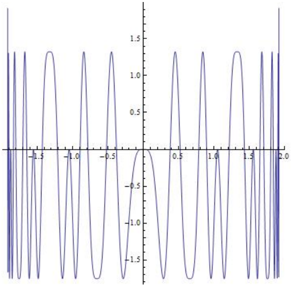

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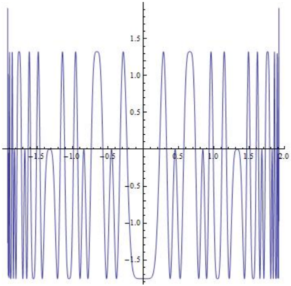

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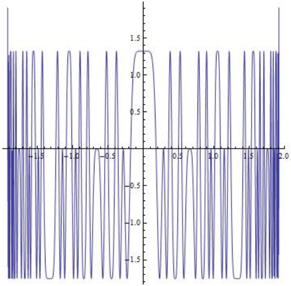

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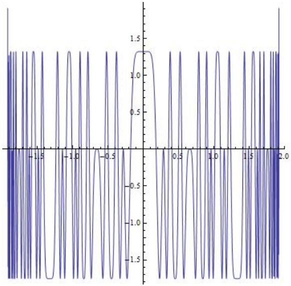
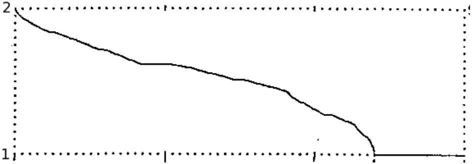


MT : Itinerary of the critical point encodes the entropy

Real quadratic polynomias

$$f_c(z) := z^2 + c, \quad c \in [-2, 1/4]$$



- $c \mapsto$entropy(f_c) is continuous (Milnor-Thurston) : kneading series
- and monotone $\log 2 \searrow 0$ (Douady, Hubbard, Sullivan).

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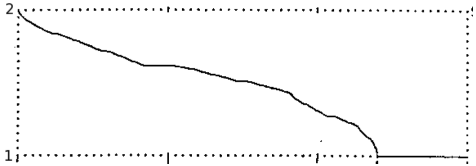

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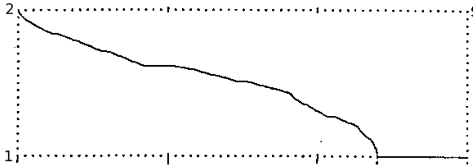

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Real polynomial acts also on $\mathbb{C}$

Julia set $K_f := \{z \in \mathbb{C} \mid f^{\circ n}(z) \nrightarrow \infty\}$.

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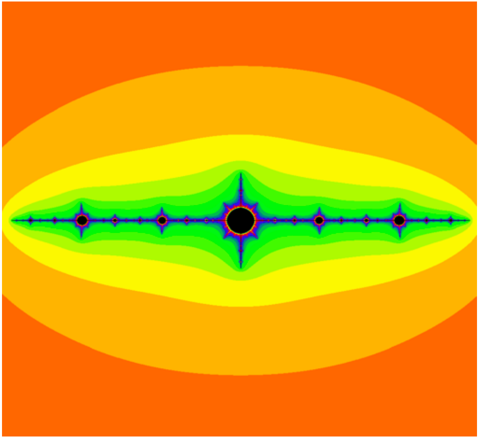
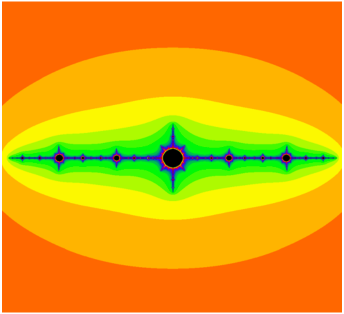
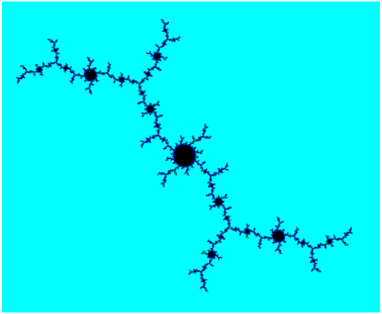


core $:= K_f \cap \mathbb{R}$, **core-entropy**$:= h_{top}(f, core)$

What is **Core**$(z^2 + c)$ for c complex?





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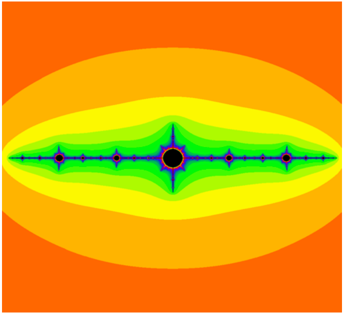

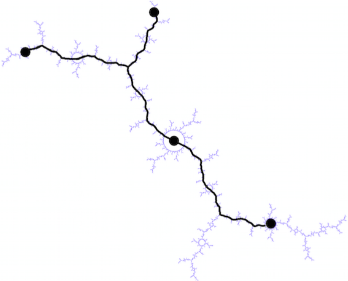

If exists, a good substitute is the Hubbard tree : the smallest finite tree in K_f containing the iterated orbit of 0. It is automatically forward invariant.

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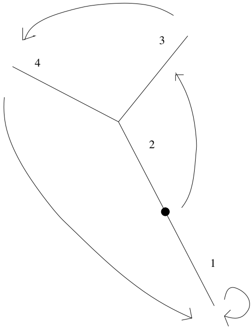

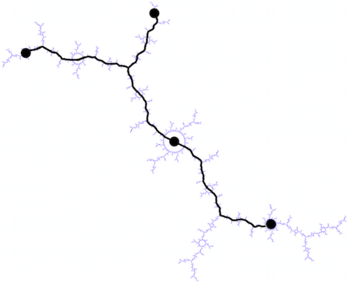

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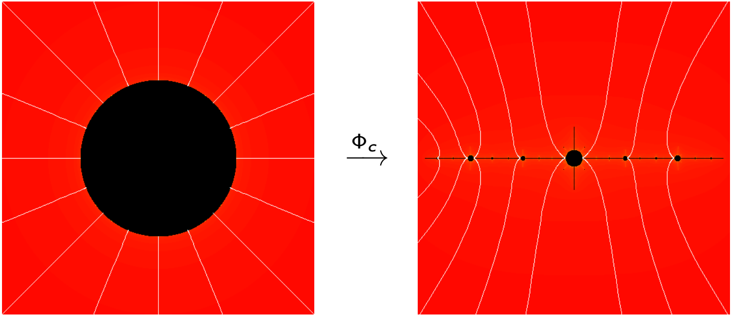
Trees do exist for a (combinatorially) dense set of $z^2 + c$...

Douady-Hubbard ray-foliation



The external rays $R_c(t)$ foliate $\mathbb{C} \setminus K_c$, and $f_c(R_c(t)) = R_c(2t)$.

Binding-entropy$:= h_{top}(F, \{\text{angle-pairs landing together}\})$,

$$F : \begin{pmatrix} s \\ t \end{pmatrix} \mapsto \begin{pmatrix} 2s \\ 2t \end{pmatrix}, \quad \mathbb{T}^2 \to \mathbb{T}^2$$

Theorem. Binding-entropy is always well defined, and $=$ Core-entropy if core exists.

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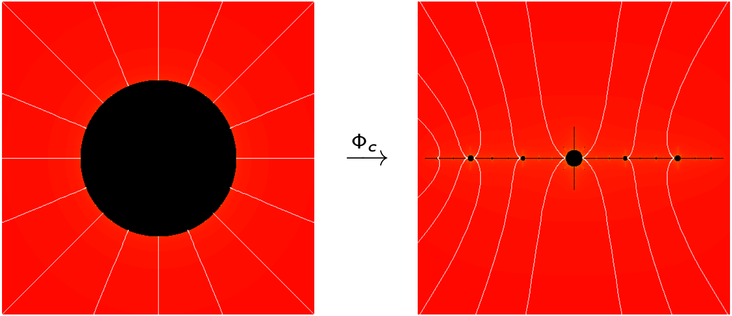

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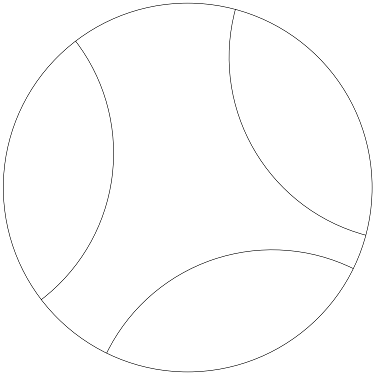
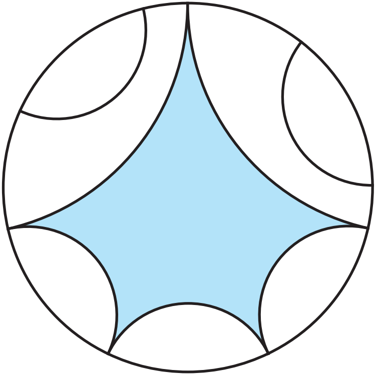
Encoding polynomials by laminations

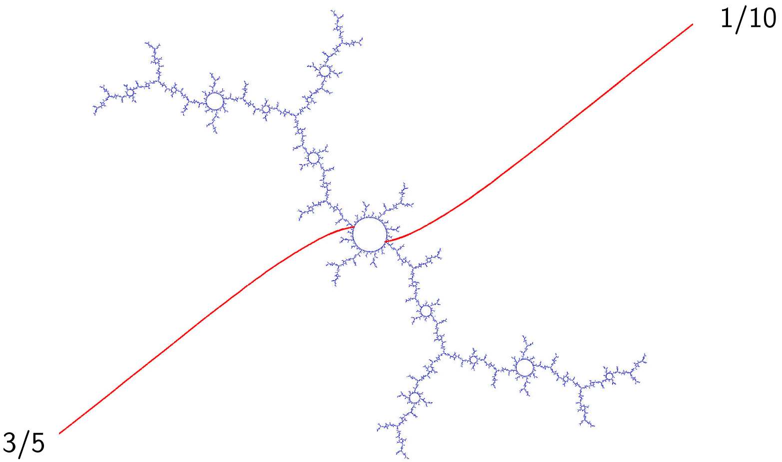
A primitive major m of degree $d =$
$\{$disjoint leaves and ideal polygons in $\mathbb{D}\}$ s.t. $\mathbb{D} \setminus m$ has d regions, each occupies $1/d$th $\partial\mathbb{D}$-space.





A degree 2 major = a diagonal chord.

The major $\{\frac{1}{10}, \frac{3}{5}\}$ encodes



Rays land at the same point iff same itinerary, or $\overline{\mathbb{D}}/\Phi_c \approx K_c$.

Combinatorial core of and its entropy

Given $d \geq 2$, consider the expanding cover

$$F : \mathbb{T}^2 \to \mathbb{T}^2, \begin{pmatrix} s \\ t \end{pmatrix} \bmod 1 \mapsto \begin{pmatrix} d \cdot s \\ d \cdot t \end{pmatrix} \bmod 1$$

Given a major m,

$$h(m) \longleftarrow \left\{ \begin{array}{ll} (\mathbb{T}^2, F) & \mathit{binding}(m) \end{array} \right.$$

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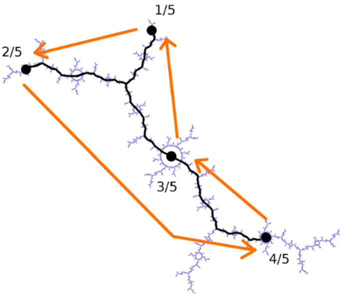
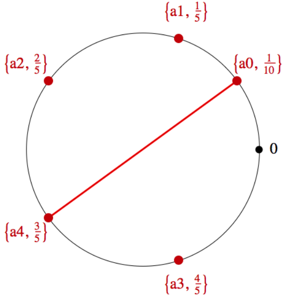
$$h(m) \longleftarrow \begin{cases} (\mathbb{C}, P_m) & \textit{core} \\ (\mathbb{T}^2, F) & \textit{binding}(m) \\ \text{if rational, } \log \lambda(\Gamma_m) \text{ by passing core} \end{cases}$$

Computing core-entropy bypassing cores (march 2011)





$$\text{basis}=\{\text{pairs}\}=\left\{\{\tfrac{1}{5},\tfrac{2}{5}\},\{\tfrac{1}{5},\tfrac{4}{5}\},\{\tfrac{1}{5},\tfrac{3}{5}\},\{\tfrac{2}{5},\tfrac{3}{5}\},\{\tfrac{2}{5},\tfrac{4}{5}\},\{\tfrac{3}{5},\tfrac{4}{5}\}\right\}$$

Linear map Γ :

$$\{\tfrac{1}{5},\tfrac{2}{5}\}\xrightarrow[\text{double angles}]{\text{same side}}\{\tfrac{2}{5},\tfrac{4}{5}\},\quad \{\tfrac{1}{5},\tfrac{4}{5}\}\xrightarrow{\text{opposite side}}\{\tfrac{1}{5},\tfrac{2}{5}\}+\{\tfrac{1}{5},\tfrac{3}{5}\}$$

$\cdots,\cdots$

Growth=1.39534. Works for any degree.

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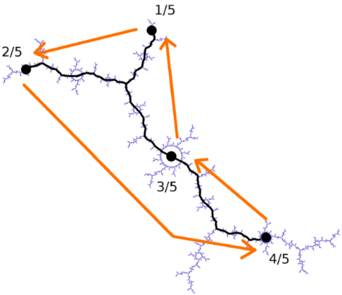

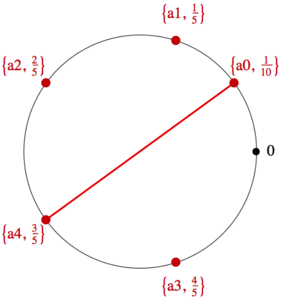

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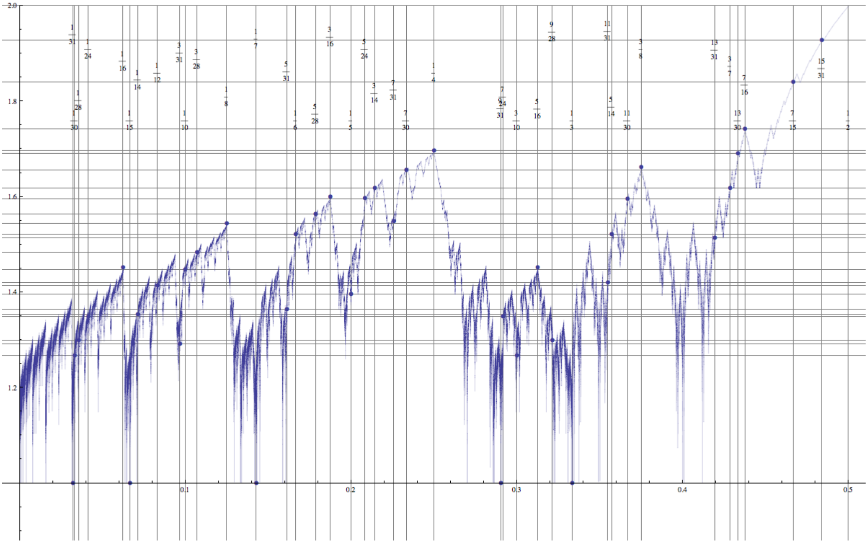
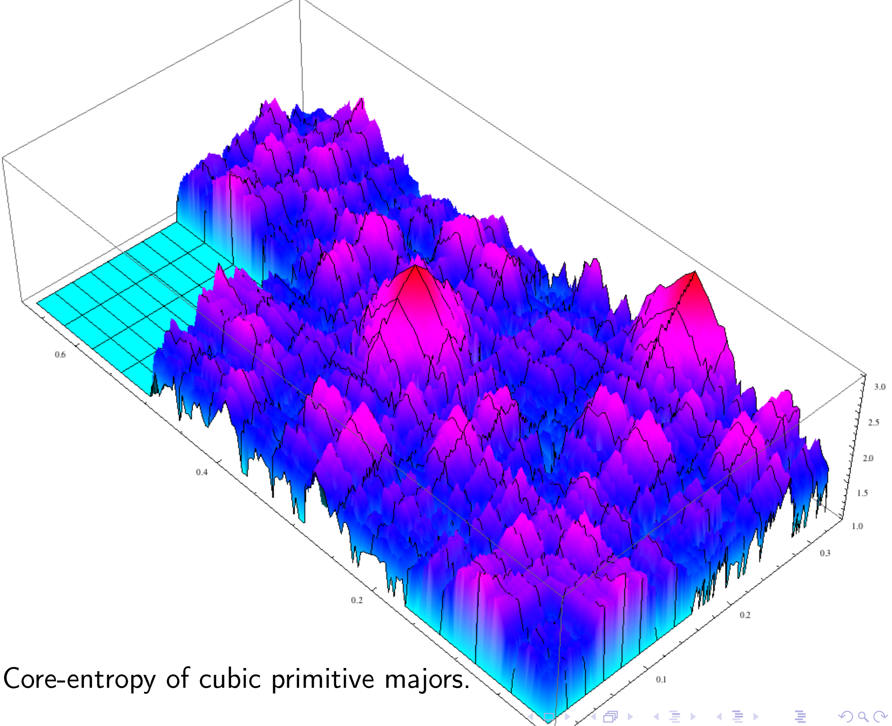
"*Jack, I'd like to share a plot I find fascinating*"



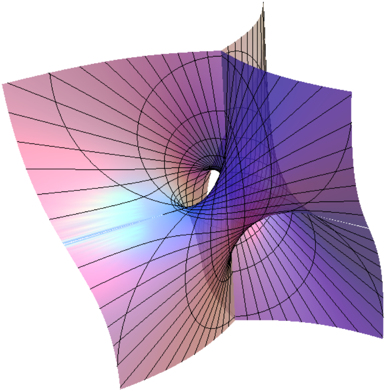


Core-entropy of cubic primitive majors.

Parametrize degree-d majors ?

Bill at Topology Festival, May 2012

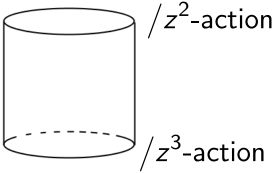
"*It took me a while to realize that the set of degree-d majors describes a spine for the set of d disjoint points in $\mathbb{C}$, that is, its fundamental group is the braid group and higher homotopy groups are trivial.*"

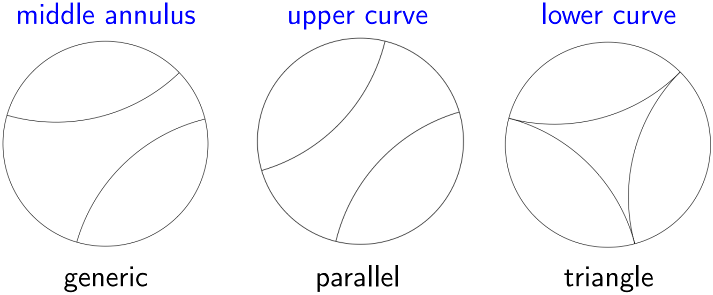


Let's try to trace Bill's discovery path : 26 march 2011

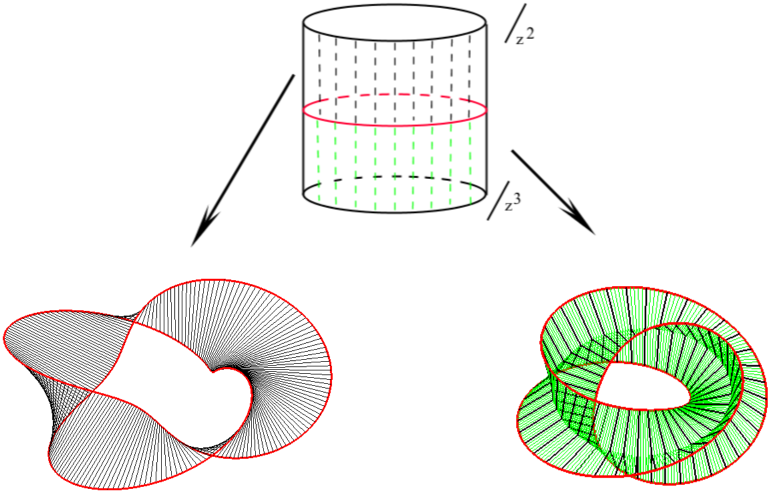
For quadratics : this just means a single diameter of the circle.

For cubics :





"*This figure can be embedded in* S^3, *which should somehow connect to the parameter space picture*"



(show Rugh's scilab animations...)

See also Branner-Hubbard 1985

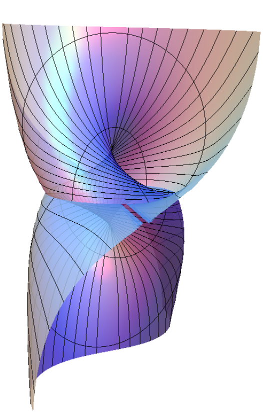
"*You can make a Moebius band by twisting a long thin strip 3 half-turns, so that its boundary is a trefoil knot. In S^3, if you make the Moebius band wider and wider, eventually you can make the boundary collide with itself in a circle, where it wraps around the circle 3 times.*

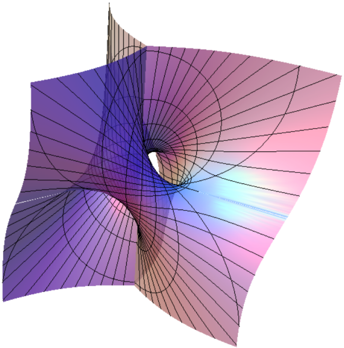
Another way to say this : take two great circles in S^3...

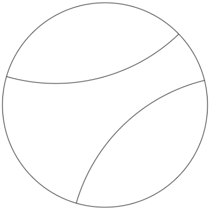
This is also a **spine** *for the complement of the discriminant locus for cubic polynomials, but I'm not sure how that description fits in.*"

Next day. "*I think you can describe this surface implicitly as ...* "

"*I don't know exactly how this relates either to the cubic major set or to the parameter space, but it's suggestive.*"



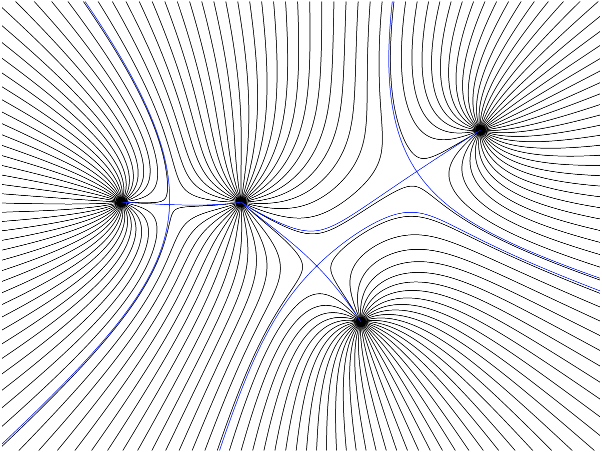




$\overset{??}{\longleftrightarrow}$ a poly. with simple roots

At last (announced 1 april 2011), a bijection !

go backward : a major $\longleftarrow$ poly. with simple roots.



$\longrightarrow$: glue slit copies of $\mathbb{C}$ following the major pattern and uniformize...

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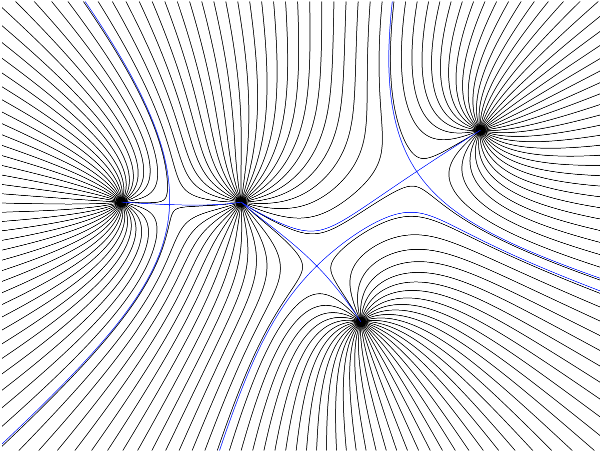

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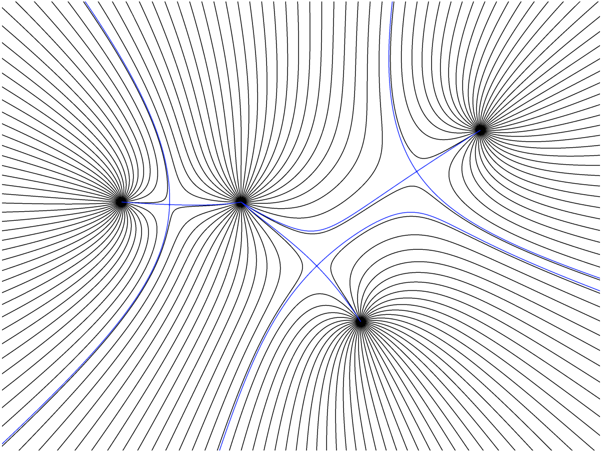

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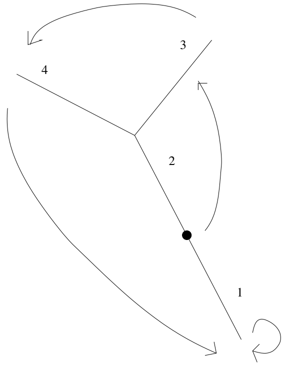
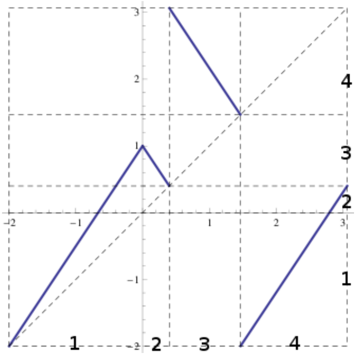
Combining with kneading theory ? Bill wrote (march 2011) :

"*For real polynomials, the kneading determinant works really well to get entropy. I'm trying to understand how this theory generalizes, particularly so entropy can be directly computed for cases that are not postcritically finite. Maybe there's a method that's essentially the same...*

I'm interested in understanding the monotonicity issue — how to make a partial order on the kneading data that makes entropy monotone."

Kneading on Hubbard trees, work of Tiozzo (2010-2011)





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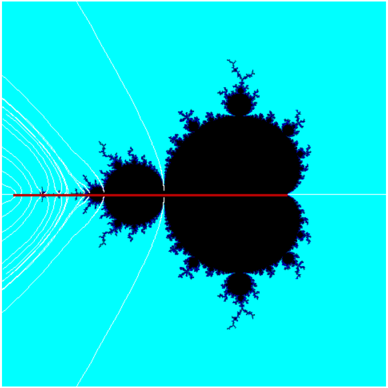
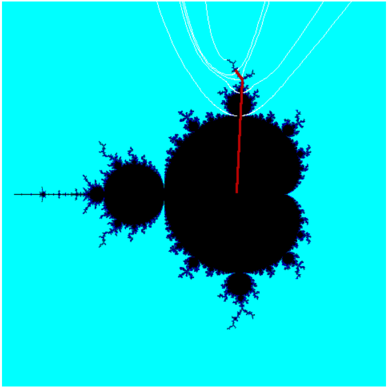




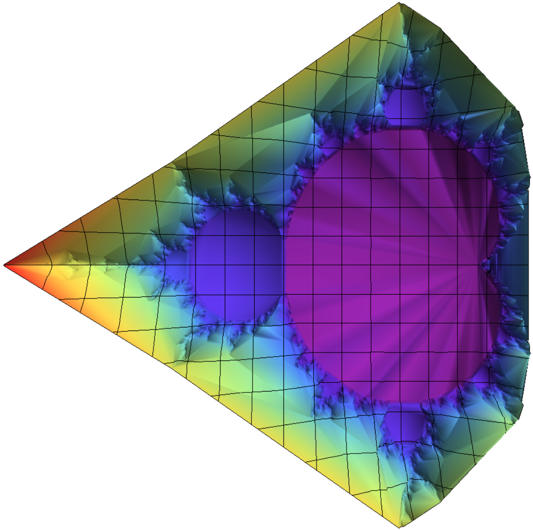
Bill in March-April 2012

"*I'm getting geared up to prove the continuity of entropy...* "

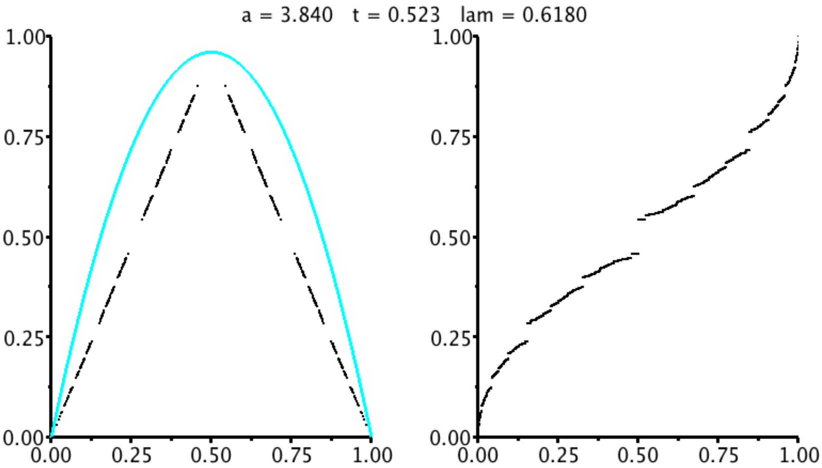
"*I've been working out more of the theory of invariant laminations, partial ordering (generalizing the tree structure for degree 2) and entropy... I've been enjoying it.* "

"*Better for more people to be involved. The next topic is to investigate more carefully the topology of the boundary of the connectedness locus for higher degree polynomials... it's rather intricate and interesting, and I want to get it right. It's a fun and interesting topic ...*"

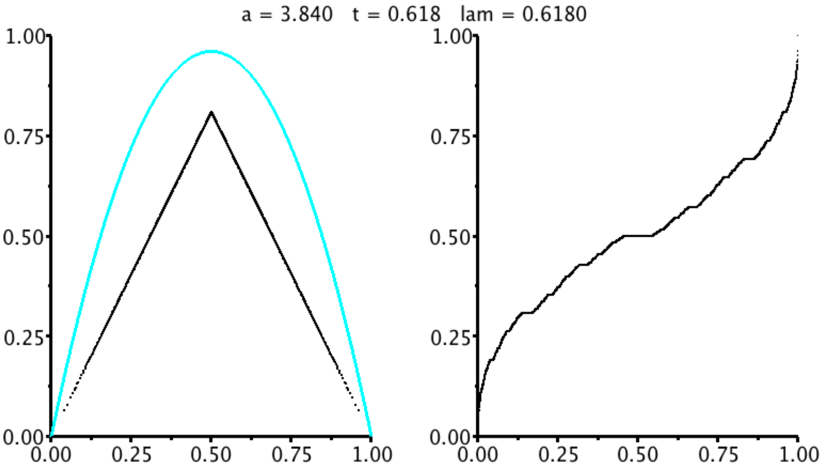
Core-entropy over M, computed by Jung, converted by Lindsey



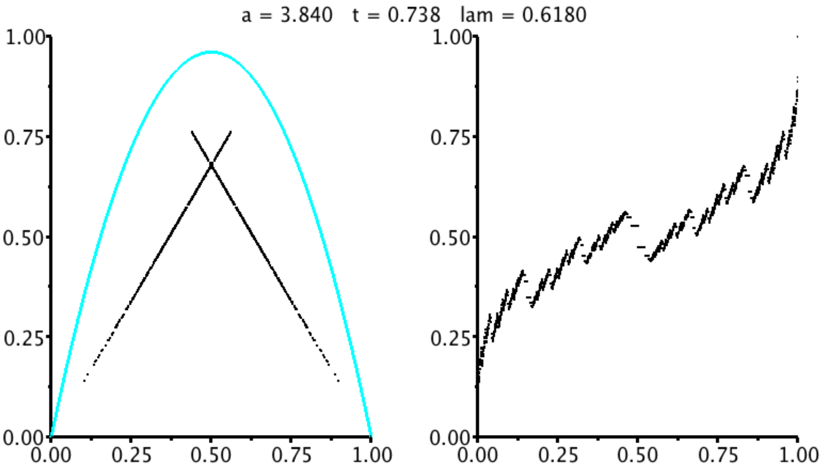
Animated Milnor-Thurston kneading theory



Animated Milnor-Thurston kneading theory



Animated Milnor-Thurston kneading theory



Contributors (incomplete list...)

Baik, Bartholdi, Chéritat, deCarvalho, Gao, Hironaka, Hubbard, Jung, Lindsey, Rugh, Dylan Thurston, Tiozzo...

Hi King Bill, Some beautiful shapes out there?