Genus minimizing knots in rational homology spheres

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"What's Next?"
The mathematical legacy of Bill Thurston
Cornell University,
June 23–27, 2014

- Thurston norm
- Heegaard Floer homology
- The rational genus bound
- $\mathbb{Z}_2$–Thurston norm and triangulations

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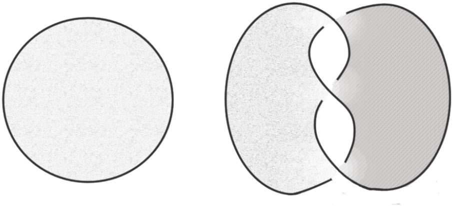
Seifert genus

The (Seifert) genus of a knot $K \subset S^3$ is defined to be

$$g(K) = \min\{g(F) | \ F \text{ is a Seifert surface for } K\}.$$



Genus bounds from the Alexander polynomial

Let

$$\Delta_K(t) = a_0 + \sum_{i=1}^{n} a_i(t^i + t^{-i})$$

be the symmetrized Alexander polynomial of a knot K, where $a_n \neq 0$.

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This bound is not always sharp. In fact, there are infinitely many nontrivial knots with $\Delta_K \equiv 1$.

Thurston Norm (Thurston, 1976)

Let S be a compact oriented surface with connected components

$$S_1, \ldots, S_n.$$

We define

$$\chi_-(S) = \sum_i \max\{0, -\chi(S_i)\}.$$

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Let M be a compact oriented 3–manifold, A be a homology class in $H_2(M;\mathbb{Z})$ or $H_2(M,\partial M;\mathbb{Z})$. The Thurston norm $x(A)$ of A is defined to be the minimal value of $\chi_-(S)$, where S runs over all the properly embedded oriented surfaces in M with $[S] = A$.

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Any Seifert surface can be regarded as a properly embedded surface in $M = S^3 \setminus \mathrm{int}(\nu(K))$, where $\nu(K)$ is a tubular neighborhood of K in S^3. Let A be a generator of $H_2(M,\partial M) \cong \mathbb{Z}$, then

$$x(A) = \begin{cases} 0, & \text{when } K \text{ is the unkot,} \\ 2g(K) - 1, & \text{otherwise.} \end{cases}$$

A semi-norm

The function x has the following basic properties:

- (Homogeneity) $x(nA) = |n| \cdot x(A)$, $n \in \mathbb{Z}$.
- (Triangle Inequality) $x(A + B) \leq x(A) + x(B)$.

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The unit ball of x is a convex polytope which is symmetric in the origin, also called the Thurston polytope.

A page from Thurston’s paper “*A norm for the homology of 3–manifolds*”

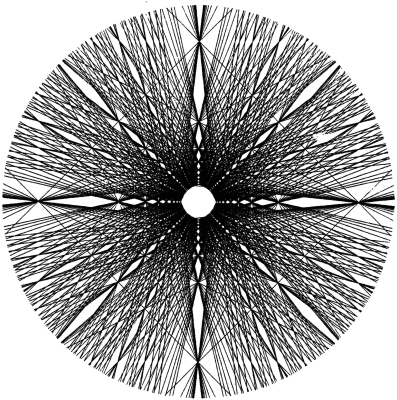

Figure 1

Lines of the form $nx + my = 1/2$ where n and m are integers. Any convex polygon in this network which is symmetric in the origin is the unit sphere in $H_2(M)$, for some 3-manifold M.

This computer drawn picture was prepared by Nathaniel Thurston.

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Thurston norm and taut foliations

Theorem (Thurston)

Suppose that M is a compact oriented 3–manifold. Let $\mathscr{F}$ be a taut foliation over M such that each component of ∂M is either a leaf of $\mathscr{F}$ or transverse to $\mathscr{F}$, and in the latter case $\mathscr{F}|\partial M$ is also taut. Then every compact leaf of $\mathscr{F}$ attains the minimal χ_- in its homology class.

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The proof uses a technique independently developed by Roussarie and Thurston (in his thesis).

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Gabai proved a converse to the above theorem.

Theorem (Gabai)

Suppose that M is a compact oriented irreducible 3*–manifold with (possibly empty) boundary consisting of tori. Let* $S \subset M$ *be a properly embedded surface which minimizes* χ_- *in the homology class of* $[S] \in H_2(M, \partial M)$*. Then there exists a taut foliation* $\mathscr{F}$ *over M such that S consists of compact leaves of* $\mathscr{F}$*.*

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$$\begin{array}{ccccc} \mathrm{Spin}^c(Y) & \times & H^2(Y) & \to & \mathrm{Spin}^c(Y) \\ \mathfrak{s} & & \alpha & \mapsto & \mathfrak{s}+\alpha. \end{array}$$

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$$c_1(\mathfrak{s}_1) - c_1(\mathfrak{s}_2) = 2(\mathfrak{s}_1 - \mathfrak{s}_2).$$

Heegaard Floer homology

Let Y be a closed, oriented, connected 3-manifold, $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. Ozsváth and Szabó defined a package of invariants associated with $(Y, \mathfrak{s})$: $\widehat{HF}(Y, \mathfrak{s}), HF^+(Y, \mathfrak{s}) \ldots$

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For each Y, there are only finitely many $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ such that $\widehat{HF}(Y, \mathfrak{s}) \neq 0$ ($\iff$ $HF^+(Y, \mathfrak{s}) \neq 0$).

Heegaard Floer homology detects the Thurston norm

Theorem (Ozsváth–Szabó)

Suppose that Y is a closed oriented 3–manifold, $A \in H_2(Y)$. Then

$$x(A) = \max\left\{ \langle c_1(\mathfrak{s}), A\rangle \,\middle|\, \mathfrak{s} \in \mathrm{Spin}^c(Y),\ HF^+(Y,\mathfrak{s}) \neq 0 \right\}.$$

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This theorem can be viewed as a generalization of McMullen's Alexander bound of the Thurston norm.

Knot Floer homology and Seifert genus

There are also versions of the previous theorem for manifold with torus boundary.

When K is a knot in S^3, its knot Floer homology is a finitely generated bigraded abelian group

$$\widehat{HFK}(K) = \bigoplus_{i,j} \widehat{HFK}_j(K, i).$$

Here i is called the "Alexander grading", and j is the "Maslov grading" or "homological grading". This invariant was introduced by Ozsváth–Szabó and Rasmussen.

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This theorem has been genralized to links in S^3 (Ozsváth–Szabó) and in arbitrary closed 3–manifold (Ni).

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Later developments allow us to bypass these contact and symplectic results (Juhász, Kronheimer–Mrowka, Ni).

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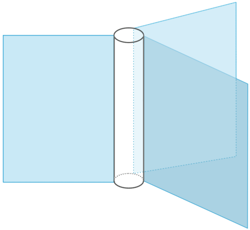
A properly embedded oriented surface $F \subset M = Y \setminus \mathrm{int}(\nu(K))$ is called a rational Seifert surface for K, if ∂F consists of coherently oriented parallel curves on ∂M, and the orientation of ∂F is coherent with the orientation of K.



Rational genus

Calegari–Gordon: The rational genus of K is defined to be

$$g_r(K) = \min_F \frac{\chi_-(F)}{2|[\mu] \cdot [\partial F]|},$$

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When K is null-homologous and nontrivial,

$$g_r(K) = \frac{2g(K) - 1}{2} = g(K) - \frac{1}{2}.$$

A function on $\operatorname{Tors} H_1(Y)$

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By definition, Turaev's lower bound is always less than 1.

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$$\begin{aligned} d(S^3, 0) &= 0, \\ d(L(p,q), i) &= -\frac{1}{4} + \frac{(2i + 1 - p - q)^2}{4pq} - d(L(q, r), j), \end{aligned}$$

where $0 \leq i < p$, r and j are the reductions modulo p of q and i, respectively.

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There are also closed formulas for $d(L(p, q), i)$ involving Dedekind sums (Némethi, Tange) or Dedekind–Rademacher sums (Jabuka–Robins–Wang).

The rational genus bound

Theorem (Ni–Wu)

Suppose that Y is a rational homology 3–sphere, $K \subset Y$ is a knot, F is a rational Seifert surface for K. Then

$$1 + \frac{-\chi(F)}{|[\partial F] \cdot [\mu]|} \geq \max_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \left\{ d(Y, \mathfrak{s} + \mathrm{PD}[K]) - d(Y, \mathfrak{s}) \right\}.$$

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The right hand side of the inequality only depends on the manifold Y and the homology class of K, so it gives a lower bound for $1 + \Theta(a)$ for the homology class $a = [K]$.

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Given a 3–manifold Z, a rationally null-homologous knot $K \subset Z$ is a Floer simple knot if

$$\operatorname{rank} \widehat{HFK}(Z,K) = \operatorname{rank} \widehat{HF}(Z),$$

where $\widehat{HFK}(Z,K)$ is the knot Floer homology of K.

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Examples of L-spaces include lens spaces, spherical space forms, double branched cover of S^3 branched along alternating links ...

Given a 3–manifold Z, a rationally null-homologous knot $K \subset Z$ is a Floer simple knot if

$$\operatorname{rank} \widehat{HFK}(Z,K) = \operatorname{rank} \widehat{HF}(Z),$$

where $\widehat{HFK}(Z,K)$ is the knot Floer homology of K.

Corollary (Ni–Wu)

The bound for Θ via correction terms is sharp for the homology classes represented by Floer simple knots in L-spaces. In fact, Floer simple knots in L-spaces attain the minimal values of the rational genus in their homology classes.

Simple knots

Let $U_1 \cup U_2$ be the genus one Heegaard splitting of $L(p, q)$. Let D_i be the meridian disk of U_i, then $\partial D_1 \cap \partial D_2$ consists of p points.

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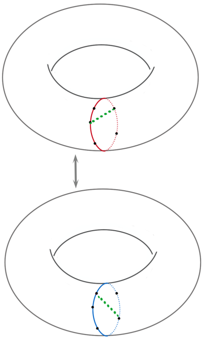
Pick any two points in $\partial D_1 \cap \partial D_2$, connecting them with arcs $\gamma_1 \subset D_1$ and $\gamma_2 \subset D_2$. The knot $\gamma_1 \cup \gamma_2$ is called a simple knot in $L(p, q)$.



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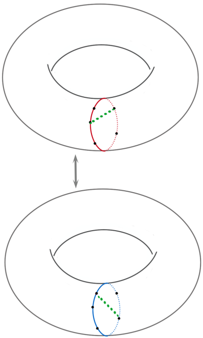

Computing Θ for lens spaces

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Our computation shows that Θ can be quite large for lens spaces. For example, in $L(p, 1)$, for the homology class $a \in \{0, 1, \ldots, p-1\}$,

$$\Theta(a) = \max\{0, \frac{a(p-a)}{p} - 1\}.$$

So if $a \sim \frac{p}{2}$, $\Theta(a) \sim \frac{p}{4}$.

Lens space surgery

Theorem (Hedden, Rasmussen)

Suppose that $L(p, q)$ is obtained by p-surgery on a knot $K \subset S^3$, then the dual knot $K' \subset L(p, q)$ is a Floer simple knot, and it is a rational genus minimizer in its homology class.

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There are similar results for lens space surgery on knots in lens spaces (studied by Boileau–Boyer–Cebanu–Walsh) or $S^1 \times S^2$ (studied by Cebanu, Baker–Buck–Lecuona).
Thus it is an interesting problem to find all the rational genus minimizers in lens spaces.

Uniqueness of genus minimizers

When $\Theta(a) < \frac{1}{2}$ and the minimal genus rational Seifert surface has only one boundary component, Baker proved that any rational genus minimizer in the homology class *a* must have bridge number 1.

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Rasmussen asked the question whether simple knots are the unique rational genus minimizers in lens spaces.

Non-uniqueness of genus minimizers

Theorem (Greene–Ni)

There are infinitely many triples (p, q, a)*, such that there are non-simple rational genus minimizers in the homology class* $a \in H_1(L(p, q))$*. Moreover, there exist infinitely many triples* (p, q, a)*, such that there are infinitely many rational genus minimizers in the homology class* $a \in H_1(L(p, q))$*.*

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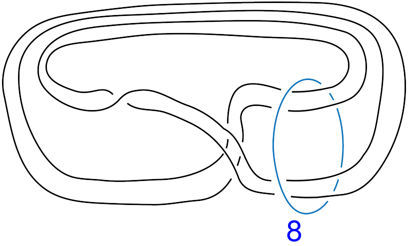
All the examples we have found have large Θ. It is possible that the uniqueness holds when Θ is small. For example, when $\Theta < \frac{1}{2}$ or even $\Theta < 1$.

The simplest example

The simplest example we have found is the $(1,2)$–cable of the $(1,2)$–torus knot in $L(8,1)$. The simple knot in this homology class is the $(1,4)$–torus knot.



- Thurston norm
- Heegaard Floer homology
- The rational genus bound
- $\mathbb{Z}_2$–Thurston norm and triangulations

Non-orientable genus

Fact: Any non-orientable surface $\Pi \subset Y$ represents a nonzero class in $H_2(Y; \mathbb{Z}_2)$. Conversely, any nonzero class in $H_2(Y; \mathbb{Z}_2)$ is represented by a non-orientable surface.

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This $h(Y, A)$ is closely related to the the so-called $\mathbb{Z}_2$-Thurston norm $||A||_{\mathbb{Z}_2}$ of A. Similar to the Thurston norm, $||A||_{\mathbb{Z}_2}$ is defined to be the minimal χ_- of (not necessarily orientable) surfaces representing A.

Non-orientable genus and Θ

When the order of $[K] \in H_1(Y;\mathbb{Z})$ is 2, any rational Seifert surface F gives rise to a closed non-orientable surface $\widehat{F} \subset Y$, such that $\beta([\widehat{F}]) = [K]$, where

$$\beta : H_2(Y;\mathbb{Z}_2) \to H_1(Y;\mathbb{Z})$$

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Proposition

Let Y be a rational homology 3–sphere. Given a nonzero class $A \in H_2(Y;\mathbb{Z}_2)$, if $h(Y,A) \geq 2$, then we have

$$h(Y,A) = 2\Theta(\beta(A)) + 2.$$

Bounding the non-orientable genus

Corollary

Let Y be a rational homology 3–sphere, $A \in H_2(Y;\mathbb{Z}_2)$, then

$$h(Y,A) \geq 2 \max_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} \left\{ d(Y,\mathfrak{s}+\mathrm{PD}\circ\beta(A)) - d(Y,\mathfrak{s}) \right\}.$$

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Levine–Ruberman–Strle proved that the bound in the above corrollary is also a lower bound to the non-orientable genus in $Y \times I$.

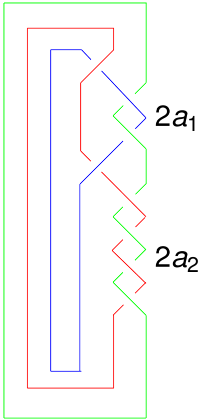
More computations

Ni–Wu:
Let L be the closure of the pure 3–braid

$$\sigma = \sigma_1\sigma_2^{-2a_1}\sigma_1\sigma_2^{-2a_2}\cdots\sigma_1\sigma_2^{-2a_{2n-1}}\sigma_1\sigma_2^{-2a_{2n}},$$

where $a_i, n > 0$, and $\Sigma(L)$ be the double branched cover of S^3 branched along L. Then the $\mathbb{Z}_2$–Thurston norms of the three nonzero homology classes in $H_2(\Sigma(L);\mathbb{Z}_2)$ are

$$\sum_{i \text{ odd}} a_i + n - 2, \quad \sum_{i \text{ even}} a_i + n - 2, \quad \sum_{i=1}^{2n} a_i - 2.$$



The complexity of 3-manifolds

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Theorem (Jaco–Rubinstein–Tillmann)

Let Y be a closed, orientable, irreducible, atoroidal, connected 3–manifold with triangulation $\mathscr{T}$. Let $H \subset H_2(Y;\mathbb{Z}_2)$ be a rank 2 subgroup, then

$$|\mathscr{T}| \geq 2 + \sum_{A \in H} ||A||_{\mathbb{Z}_2}.$$

The theorem of Ni–Wu implies that $||\cdot||_{\mathbb{Z}_2}$ is bounded below in terms of correction terms. As a result, $C(Y)$ is bounded below in terms of correction terms in the cases discussed in Jaco–Rubinstein–Tillmann.

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On the other hand, we can construct a triangulation of $\Sigma(L)$ with

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tetrahedra. So we bound $C(\Sigma(L))$ in a range of length $2n+4$. We should be able to do much better according to Rubinstein.

Thank you!