Thurston's theorems in complex dynamics

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What's Next? The mathematical legacy of Bill Thurston
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Thurston's theorem on the characterization of rational maps among self-branched covering of 2-sphere.

Let $f : S^2 \to S^2$ be a branched covering with PCF condition ("finiteness"). Then f is equivalent ("conjugacy up to isotopy") to a rational map if and only if there are no "Thurston obstructions". The obstruction is characterized by a collection of simple closed curves and the action of f on them.

Formulation via Teichmüller space and idea of proof.

Applications.

Reduction to finitely checkable statements.
Understanding the parameter space of $z^2 + c$ (Mandelbrot set).
Rigidity, Monotonicity.
Matings.

Some developments around the theorem.

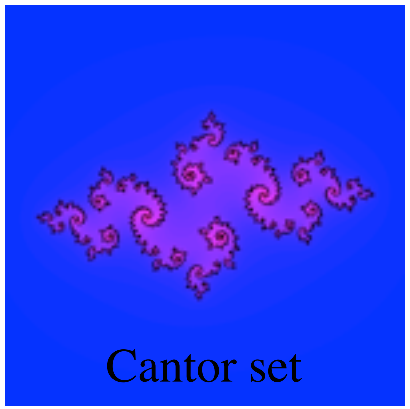
What's next?

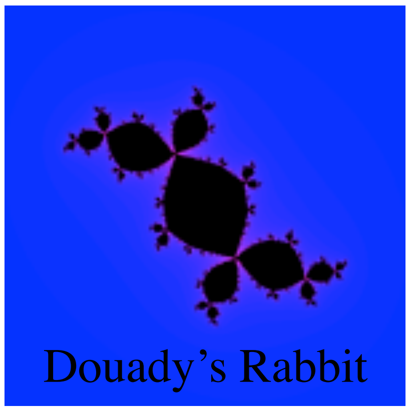
Understanding Dynamical Systems

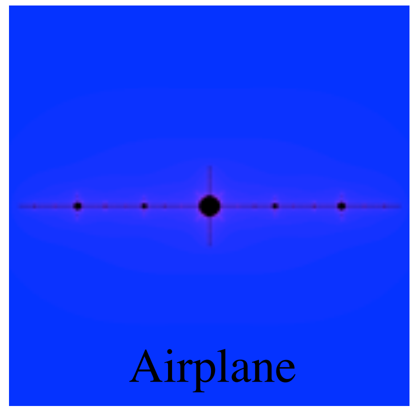
Q. Given c, what is the dynamics of $f_c : z \mapsto z^2 + c$?
(e.g. topology of Julia sets, etc.)

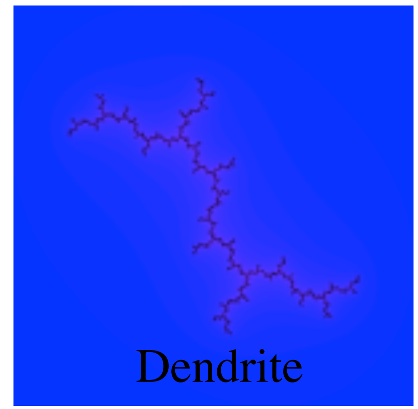
On $[0, 1]$, with $a = 3.8$, consider $x \mapsto ax(1 - x)$.

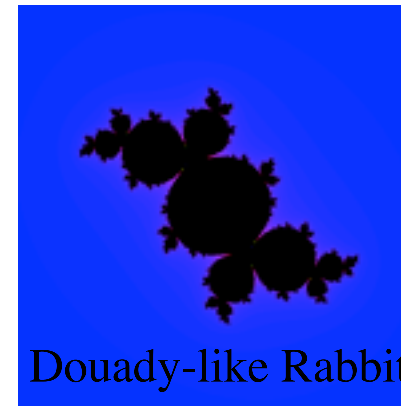
Sometimes we can answer to this question. (but NOT always)


Cantor set


Douady's Rabbit


Airplane


Dendrite


Douady-like Rabbit

If a rational map is *hyperbolic* (all critical points are attracted to attracting periodic orbits), then we understand completely. (Top. Structure, Structural stability, measure of Julia sets, etc.)

It is important to ask the right questions.

Example: What is the possible dynamics among $z \mapsto z^2 + c$?

What's next?

Formulation

Let $f : S^2 \to S^2$ be a branched covering. (locally like $z \mapsto z^k$)

$Crit(f) = \{\text{critical pts of } f\}$, $P_f = \bigcup_{n=1}^{\infty} f^n(Crit(f))$ (post-crit. set)

Assume $\sharp P_f < \infty$. (*Post-critically finite*, PCF)

Two PCF branched coverings f and g are *equivalent*, $f \sim g$, if there exist two orientation preserving homeomorphisms $\theta_1, \theta_2 : S^2 \to S^2$ such that

$\theta_i(P_f) = P_g$ $(i = 1, 2)$, $\theta_1 = \theta_2$ on P_f, θ_1 and θ_2 are isotopic relative to P_f, and the following diagram commutes:

$$\begin{array}{ccc} S^2 & \xrightarrow{\theta_1} & S^2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ S^2 & \xrightarrow{\theta_2} & S^2. \end{array}$$

Q. Given f as above, when is it equivalent to a rational map?

We start from the idea of the proof, rather than the statement of the theorem.

Reformulating the question in terms of Teichmüller space

S_* oriented hyperbolic surface. The *Teichmüller space* is

$Teich(S_*) = \{(S,\theta) \mid S \text{ Riemann surface}, \ \theta : S_* \to S \text{ o.-p. homeo}\}/\sim$

$(S_1,\theta_1) \sim (S_2,\theta_2) \iff \theta_2 \circ \theta_1^{-1} : S_1 \to S_2$ is isotopic to a conformal map.

complex structure, complete w.r.t. the Teichmüller distance $d(\cdot,\cdot)$

Let $f : S^2 \to S^2$ be PCF branched covering. Suppose $n = \sharp P_f \geq 5$.

Take $S_* = S^2 \setminus P_f$ or S^2 with n marked points.

Define $\sigma_f : Teich(S_*) \to Teich(S_*)$ by $\sigma_f([\theta_1]) = [\theta_2]$, where

$$\begin{array}{ccc} S^2 & \xrightarrow{\theta_2} & \widehat{\mathbb{C}} \\ f\Big\downarrow & & \Big\downarrow F_{[\theta_1]} \\ S^2 & \xrightarrow{\theta_1} & \widehat{\mathbb{C}} \end{array}$$

holomorphic (rational map)

So the previous question is equivalent to:

Q. Does σ_f has a fixed point?

"Always look for a fixed point"

compare: classification of surface homeos

Royden's theorem. (Teichmüller space version of Schwarz Lemma)
Any holomorphic mapping between Teichmüller spaces is either isometric or non-uniformly contracting.

compare: skinning map for the hyperbolization of 3-manifolds

We can show directly that σ_f in our setting is non-uniformly contracting. And the contraction is uniform on $\pi^{-1}(K)$, where $\pi : Teich(S_*) \ni [S, \theta] \mapsto [S] \in Moduli(S_*)$ and $K \subset Moduli(S_*)$ compact.

(In fact, the "cotangent space" of $Teich(S_*)$ is identified with the space of integrable holomorphic quadratic differentials $\{q = \varphi(z)dz^2\}$. And the "cotangent map" of σ_f is the push-forward $F_*(q)$. $|q| = |\varphi(z)|dxdy$ is a measure and $\iint F_*(|q|) = \iint |q|$. Hence except in a very special situation, there is a cancellation and $||F_*(q)|| < ||q||$.)

Conclusion:
Either (A) σ_f has a fixed point. (hence $f \sim \exists$ rat map)
Or (B) For any $\tau \in Teich(S_*)$, $\sigma_f^n(\tau) \to \partial Teich(S_*)$. Moreover $\pi(\sigma_f^n(\tau)) \to \partial Moduli(S_*)$.

Problem: Characterize Case (B).

Implementing $\sigma_f : Teich(S_*) \to Teich(S_*)$

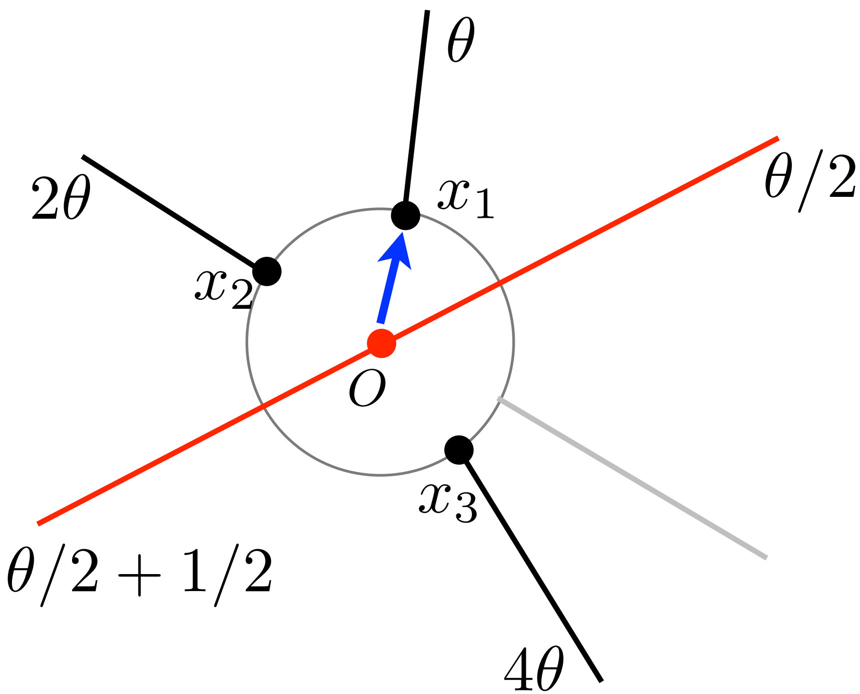
Spider algorithm: (Hubbard-* for Classic MacOS, paper: Hubbard-Schleicher)

$f : S^2 \to S^2$ deg 2 topological polynomial ($\infty \in S^2$ with $f^{-1}(\infty) = \{\infty\}$).

$\infty \in P_f, \quad P_f \setminus \{\infty\} = \{x_1, \ldots, x_n\}$

$Teich(S_*) \ni [S, h] \longleftrightarrow n$ distinct points on $\mathbb{C}$ together with isotopy classes of disjoint arcs joining x_k's to ∞.

Specify the map f by a rational angle $\theta \in \mathbb{Q}/\mathbb{Z}$.
θ is pre periodic under $t \mapsto 2t \bmod \mathbb{Z}$.



Divide the plane by the line joining $\theta/2$ and $\theta/2 + 1/2$. The inverse image of each leg has a component in each side.
According to the side to which $2^k\theta$ belong to, choose one of inverse legs of x_{k+1}.

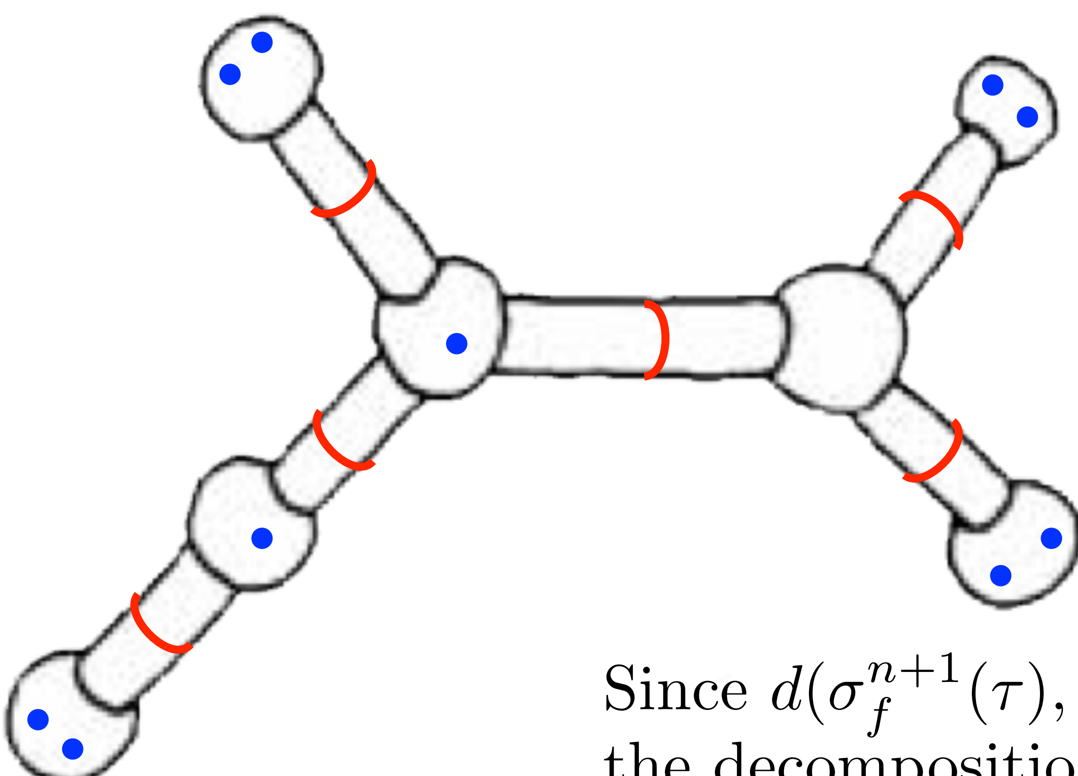
The legs converge to the external rays (and internal rays within the basins).
The angle θ corresponds to the external angle of the critical value.

Characterizing the obstruction

Either (A) σ_f has a fixed point. (hence $f \sim \exists$ rat map)
Or (B) For any $\tau \in Teich(S_*)$, $\sigma_f^n(\tau) \to \partial Teich(S_*)$. Moreover $\pi(\sigma_f^n(\tau))$ $\to \partial Moduli(S_*)$.

Problem: Characterize Case (B).

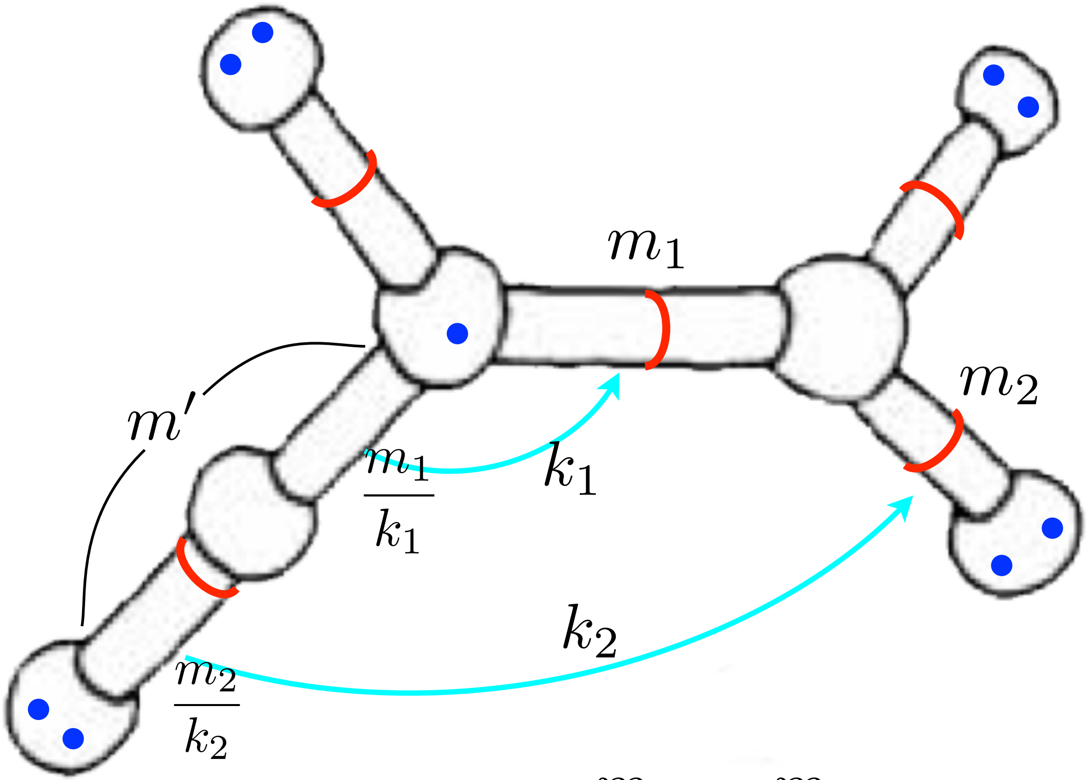
$\pi(\sigma_f^n(\tau))$ decomposes into multiple spheres joined by cylinder = tubes = annuli. Each cylinder is represented by a core curve. They form a multicurve, which is a collection of disjoint simple closed curves, ech of which is not null-homotopic nor homotopic to a puncture.



Since $d(\sigma_f^{n+1}(\tau), \sigma_f^n(\tau)) \leq d(\sigma_f(\tau), \tau)$, the decomposition or the multicurve stabilizes.

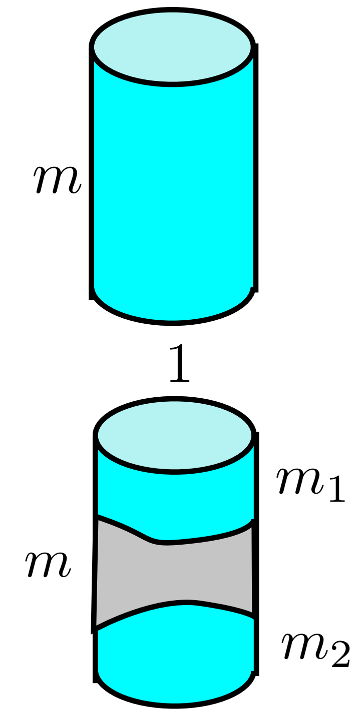
Characterizing the obstruction 2

Let Γ be an invariant multicurve. To each $\gamma \in \Gamma$, associate an annulus A_γ and its modulus m_γ.



$$m' \geq \frac{m_1}{k_1} + \frac{m_2}{k_2}$$

Modulus of annulus



$mod(A) = m$

Grötzsch inequality
$m \geq m_1 + m_2$

$f : A_1 \to A_2$ covering of deg k
$mod(A_2) = k\, mod(A_1)$

Define $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ by $f_\Gamma : (m_\gamma)_{\gamma\in\Gamma} \mapsto (m'_\gamma)_{\gamma\in\Gamma}$

where $m'_\gamma = \sum_{\delta\in\Gamma} \sum_{\substack{\delta'\subset f^{-1}(\delta)\\ \delta'\sim\gamma}} \frac{m_\delta}{deg(f:\delta'\to\delta)}$.

Define $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ by $f_\Gamma : (m_\gamma)_{\gamma\in\Gamma} \mapsto (m'_\gamma)_{\gamma\in\Gamma}$ Thurston matrix

where $m'_\gamma = \sum_{\delta\in\Gamma} \sum_{\substack{\delta' \subset f^{-1}(\delta) \\ \delta' \sim \gamma}} \frac{m_\delta}{deg(f : \delta' \to \delta)}$.

$\lambda_\Gamma =$ leading eigenvalue of f_Γ (Thurston eigenvalue)

Claim. Case (B) occurs $\Longleftrightarrow$ There exists Γ with $\lambda_\Gamma \geq 1$.
(Thurston obstruction)

If $\lambda_\Gamma > 1$, it is easy to show that $\sigma_f^n(\tau) \to \partial Teich(S_*)$. If $\lambda_\Gamma = 1$, need to consider "maximal annuli" and still get $\sigma_f^n(\tau) \to \partial Teich(S_*)$. More work is needed to show that if $\lambda_\Gamma < 1$ for any invariant multicurve, then there is a fixed point of σ_f.

Theorem (Thurston). (Published by Douady-Hubbard 1993)
A PCF branched covering $f : S^2 \to S^2$ (with $\sharp P_f \geq 5$) is equivalent to a rational map if and only if it has no Thurston obstruction, i.e., any invariant multicurve $\Gamma \subset S^2 \setminus P_f$ satisfies $\lambda_\Gamma < 1$.
Moreover when this condition holds, the equivalent rational map is unique. (Rigidity)

Applications

Construct rational maps from branched coverings. Need to check the non-existence of Thurston obstructions, this means checking for infinitely many multicurves.

A multicurve $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{p-1}\}$ is called a *Levy cycle* if each γ_i has an inverse image $\gamma'_{i-1} \subset f^{-1}(\gamma_i)$ which is homotopic to γ_{i-1} with $deg(f : \gamma'_{i-1} \to \gamma_i) = \pm 1$ ($i = 1, \ldots, p$ with $\gamma_p = \gamma_0$). By taking inverse images, Γ can be extended to be a Thurston obstruction.

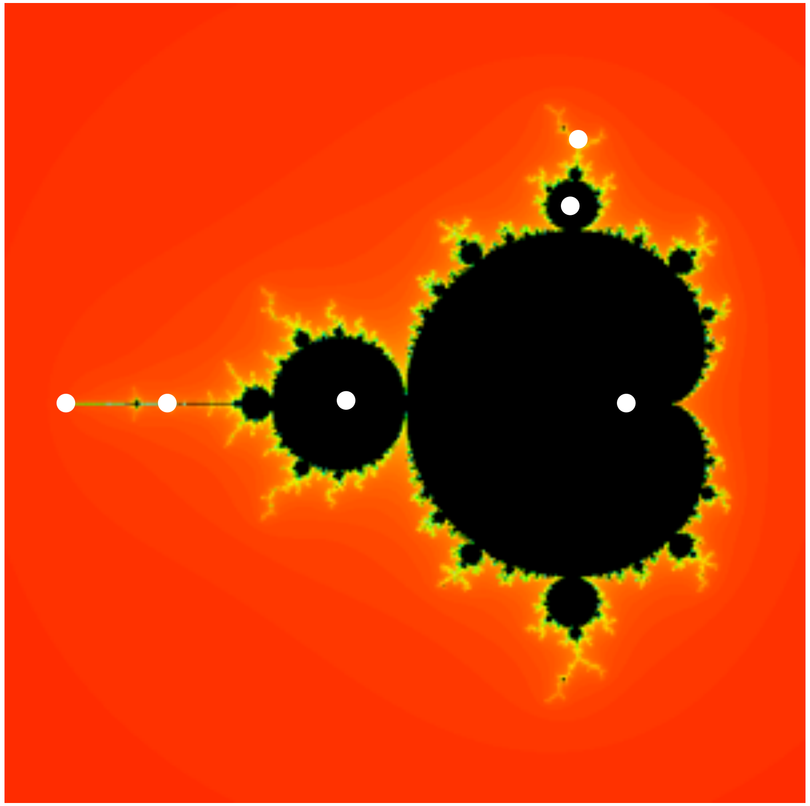
Levy cycle theorem. (Levy, Rees) If f is a topological polynomial or a branched covering of degree 2, then f has a Thurston obstruction if and only if it has a Levy cycle.

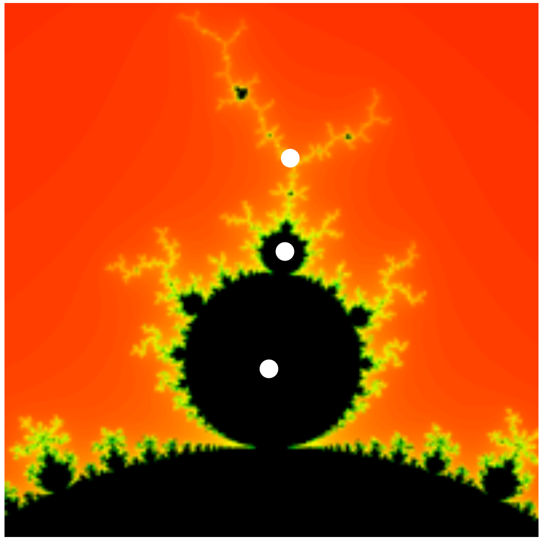
Usually Levy cycles are much easier to relate to finitely checkable combinatorial conditions.

For polynomials, one can start from Hubbard trees or Spiders and construct branched coverings and check the non-existence of Thurston obstructions. (Douady-Hubbard for quadratic case, Poirier for all degree)

Quadratic polynomials and the Mandelbrot set

$M = \{c \in \mathbb{C} |$ Julia set $J(z^2 + c)$ is connected$\}$





Douady-Hubbard gave a combinatorial description of the Mandelbrot set in terms of the PCF parameters.

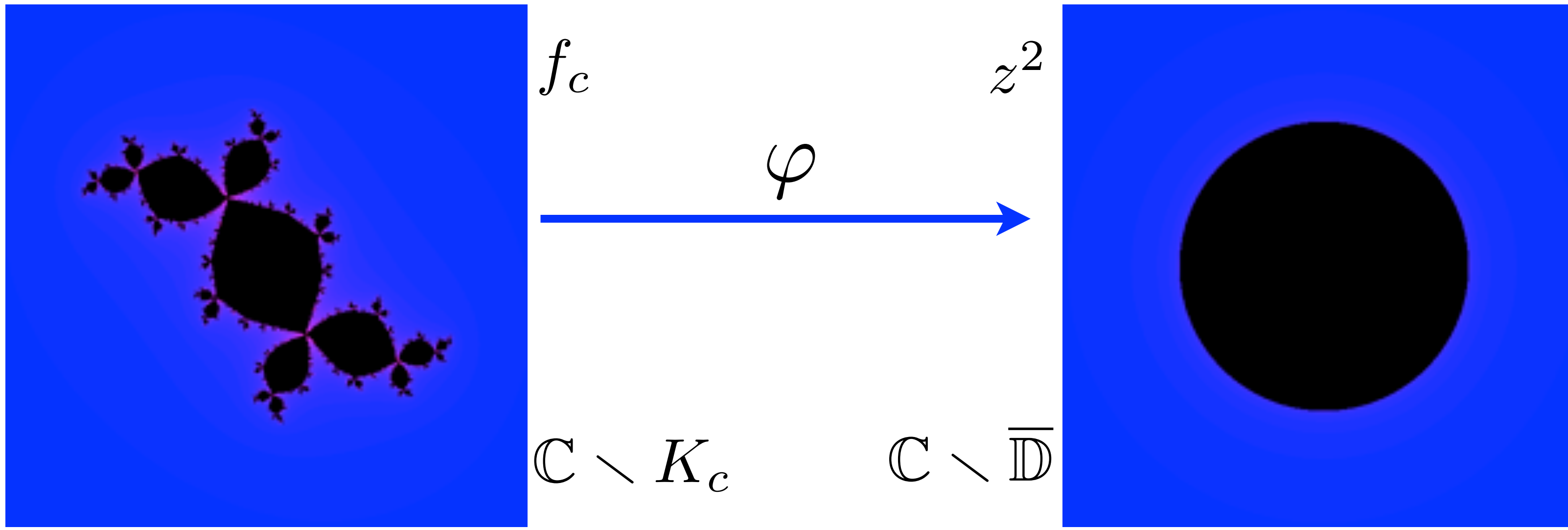
Monotonicity Theorem for real quadratic polynomials. (Milnor-Thurston, Douady-Hubbard, Sullivan) The entropy of the real map $x \mapsto ax(1 - x)$ is monotone with respect to the parameter a.

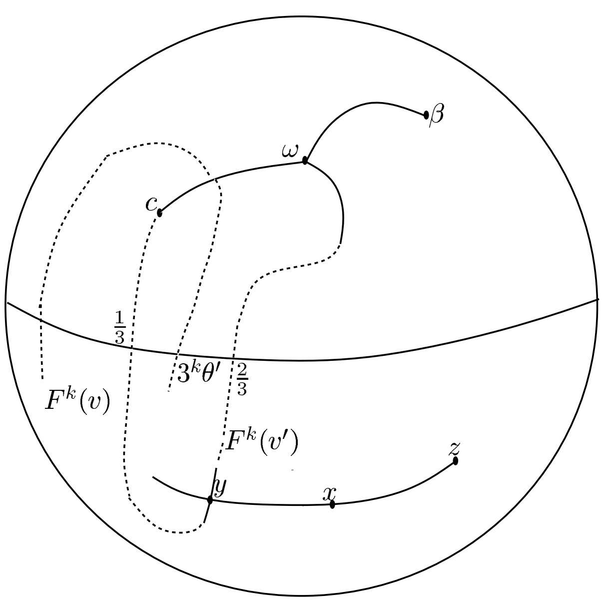
Q. Monotonicity for $x \mapsto |x|^d + c$ for $d \notin 2\mathbb{N}$?

Mating

Mating is a way to construct a (non-polynomial) branched covering from a pair of (PCF) polynomials f and g of degree $d \geq 2$.

Böttcher coordinate



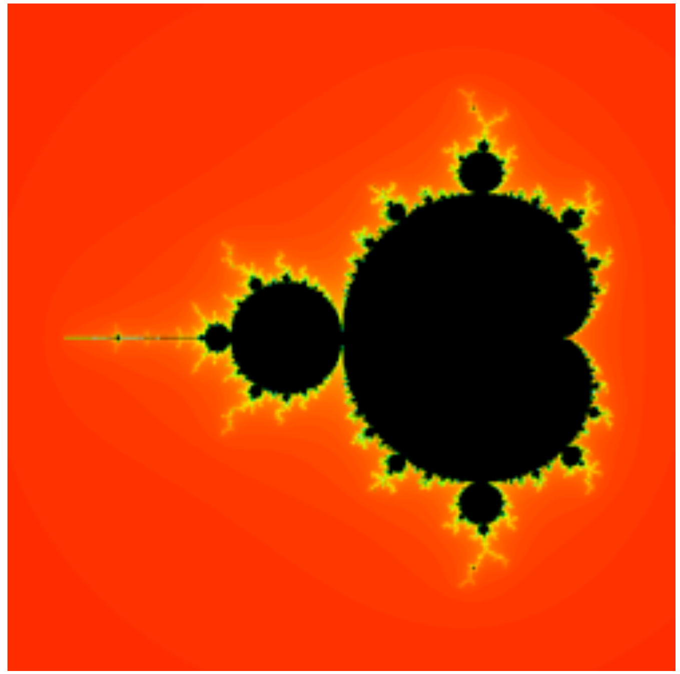


Blow up ∞ to a circle at infinity.
Then glue the two circles at infinity by the Böttcher coordinates.
The induced map $f \amalg g$ is called *formal mating*.

Q. Is $f \amalg g$ is equivalent to a rational map?

Q. If so, what is the relation between the realizing rational map and $f \amalg_{top} g$ on $K_f \sqcup K_g / \sim$ (topological mating).

Theorem. (Rees, Tan) $d = 2$. Two PCF quadratic polynomials $z^2 + c_1$ and $z^2 + c_2$ are matable ($(z^2 + c_1) \amalg (z^2 + c_2)$ is equivalent to a rational map) if and only if they do not belong to conjugate limbs of M.



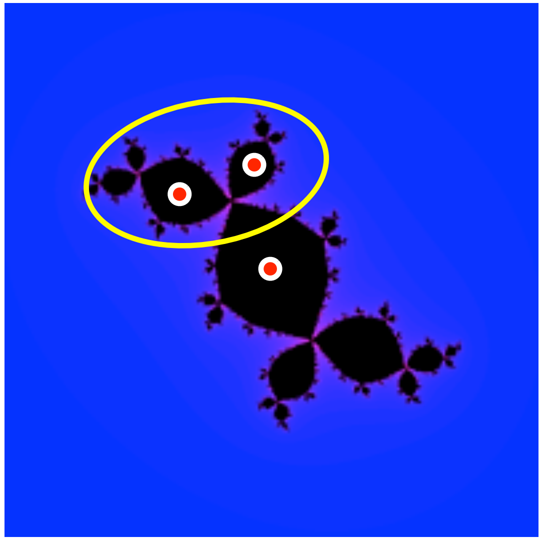
Theorem. (Rees, S.) If two PCF polynomials f and g are matable and $f \amalg g \sim R$ (after a minor modification), then the topological mating $f \amalg_{top} g$ on $K_f \sqcup K_g / \sim$ is conjugate to R.

Example. (S.-Tan) There exists a formal mating of two cubic polynomials which has a Thurston obstruction but not a Levy cycle. Moreover the topological mating defines a branched covering of S^2.

Q. Find an algorithm to check the matability.

Further developments

Thurston's theorem for topological exponential maps. Levy cycle theorem. (Hubbard-Schleicher-S.)



Twisted rabbit problem.
(Solution by Bartholdi-Nekrashevych via Iterated Monodromy Group)

Postcriticall finite maps on $\mathbb{P}^N$ induced by σ_f^{-1} on the moduli space. (Koch)

"Positive criterion" (sufficient condition) for breached covering to be equivalent to a rational map. (Dylan Thurston)

What's next?

How to detect Thurston obstructions?

  Algorithm? (cf. Yampolsky's computability result)

    Algorithm to decide matability, equivalence, shared mating etc.

  Dylan's approach using embedded graph and energy

  Intersection with a reference multicurve (non-obstruction) (S.)

Estimate the distance to the fixed point of σ_f when exists.

  Speed of convergence or rate of contraction?

  Can one prove the density of hyperbolicity via σ_f?

Conjecture. Hyperbolic maps are dense in the space of all rational maps or polynomials of degree d.

How do PCF polynomials/rational maps fit within the parameter space.

  Build a combinatorial model of the parameter space?

  Maps with different number of post critical points may be close to each other in the parameter space.

  Find the image of a subset of the Mandelbrot set within the parameter space of matings.