Lyapunov exponents of the Hodge bundle and diffusion in periodic billiards

Anton Zorich

"WHAT'S NEXT?"
THE MATHEMATICAL LEGACY OF BILL THURSTON

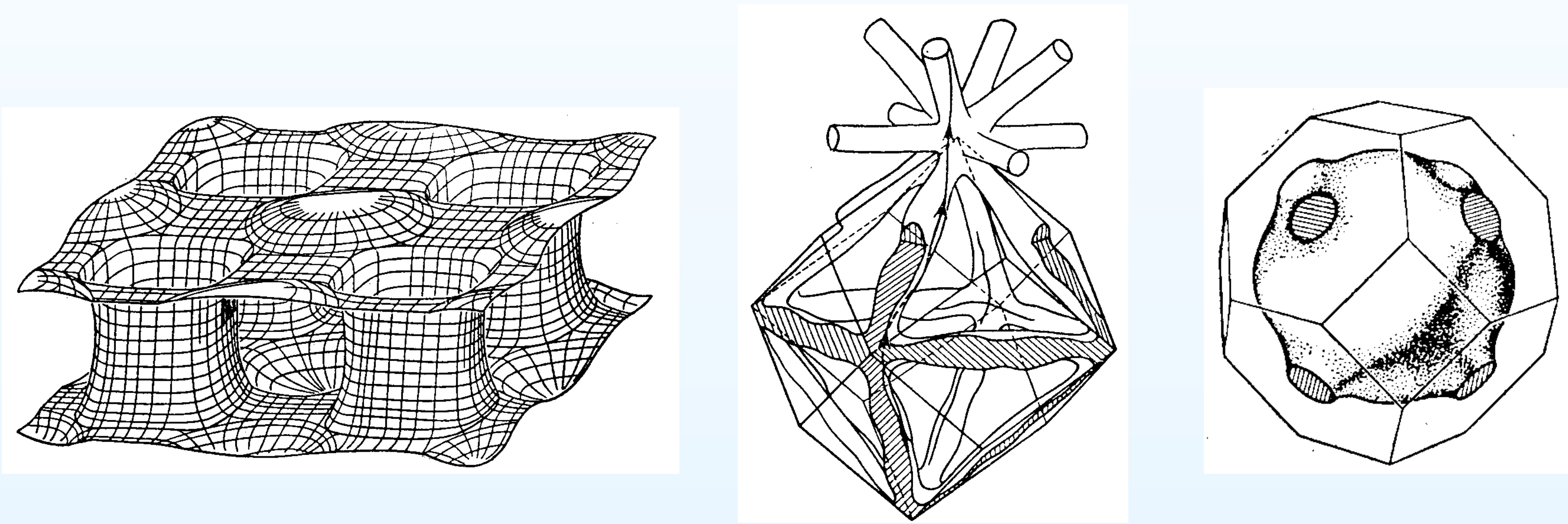
Cornell, June 24, 2014

0. Model problem: diffusion in a periodic billiard
● Electron transport in metals in homogeneous magnetic field
● Diffusion in a periodic billiard (“Windtree model”)
● Changing the shape of the obstacle
● From a billiard to a surface foliation
● From the windtree billiard to a surface foliation
1. Teichmüller dynamics (following ideas of B. Thurston)
2. Asymptotic flag of an orientable measured foliation
3. State of the art
∞. What’s next?

0. Model problem: diffusion in a periodic billiard

Electron transport in metals in homogeneous magnetic field

Measured foliations on surfaces naturally appear in the study of conductivity in crystals. For example, the energy levels in the quasimomentum space (called *Fermi-surfaces*) might give sophisticated periodic surfaces in $\mathbb{R}^3$.



Fermi surfaces of tin, iron, and gold.

Electron trajectories in the presence of a homogeneous magnetic field correspond to sections of such a periodic surface by parallel planes. Passing to the quotient by $\mathbb{Z}^3$ we get a measured foliation on the resulting compact surface.

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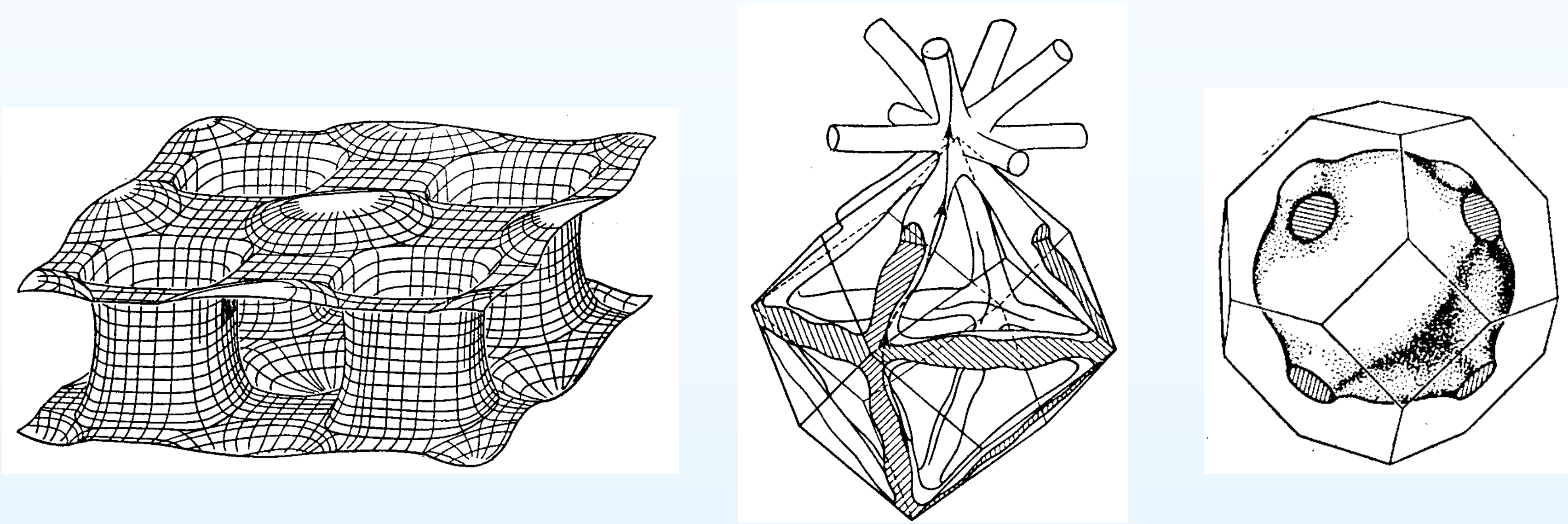

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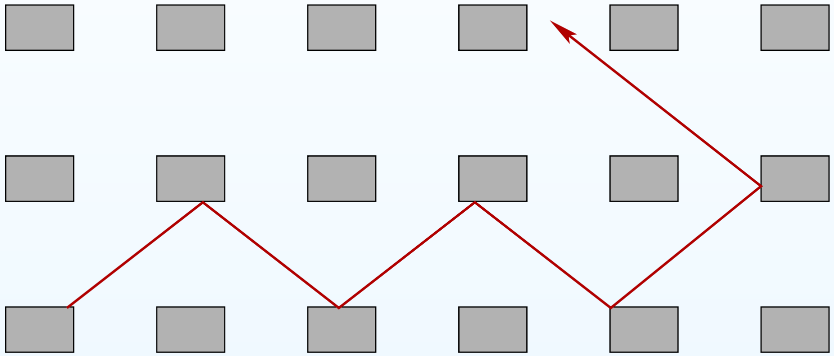

Old Theorem (V. Delecroix, P. Hubert, S. Lelièvre, 2011). *For almost all parameters of the obstacle, for almost all initial directions, and for any starting point, the billiard trajectory escapes to infinity with the rate $t^{2/3}$. That is,*

$$\max_{0\le\tau\le t}\text{(distance to the starting point at time }\tau) \sim t^{2/3}.$$

Here "$\frac{2}{3}$" is the Lyapunov exponent of certain "renormalizing" dynamical system associated to the initial one.

Remark. Changing the height and the width of the obstacle we get quite different billiards, but this does not change the diffusion rate!

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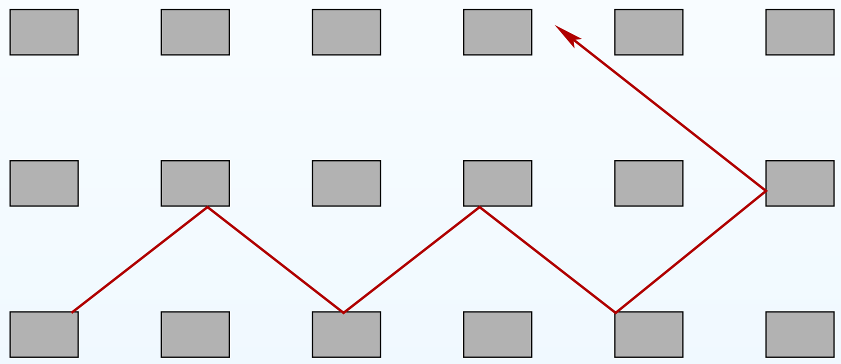

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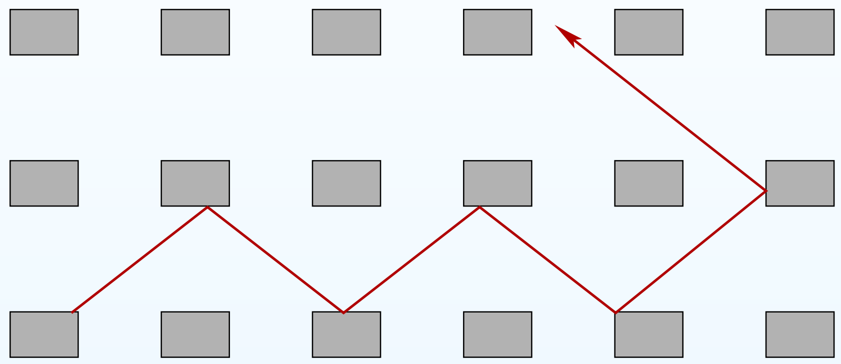

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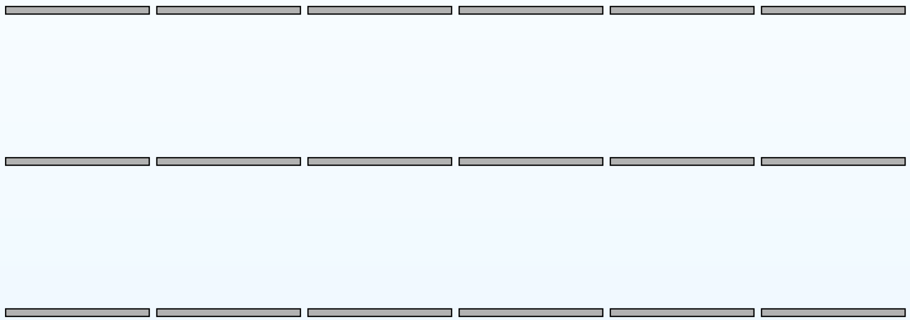

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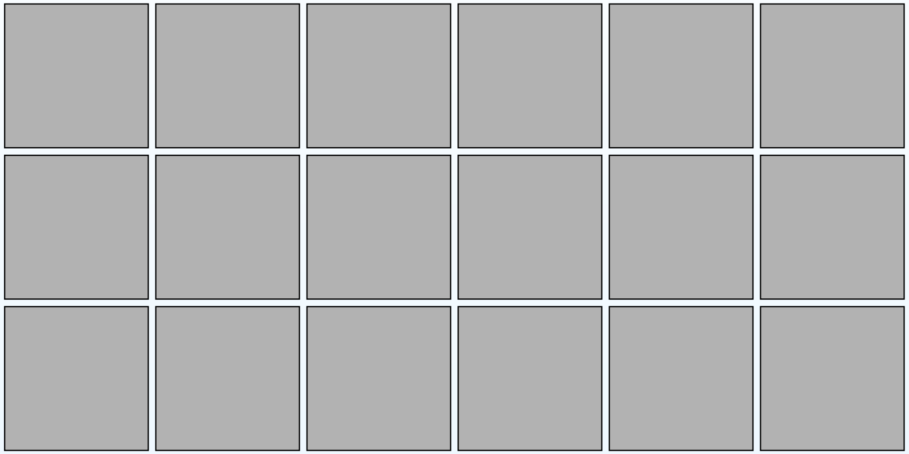

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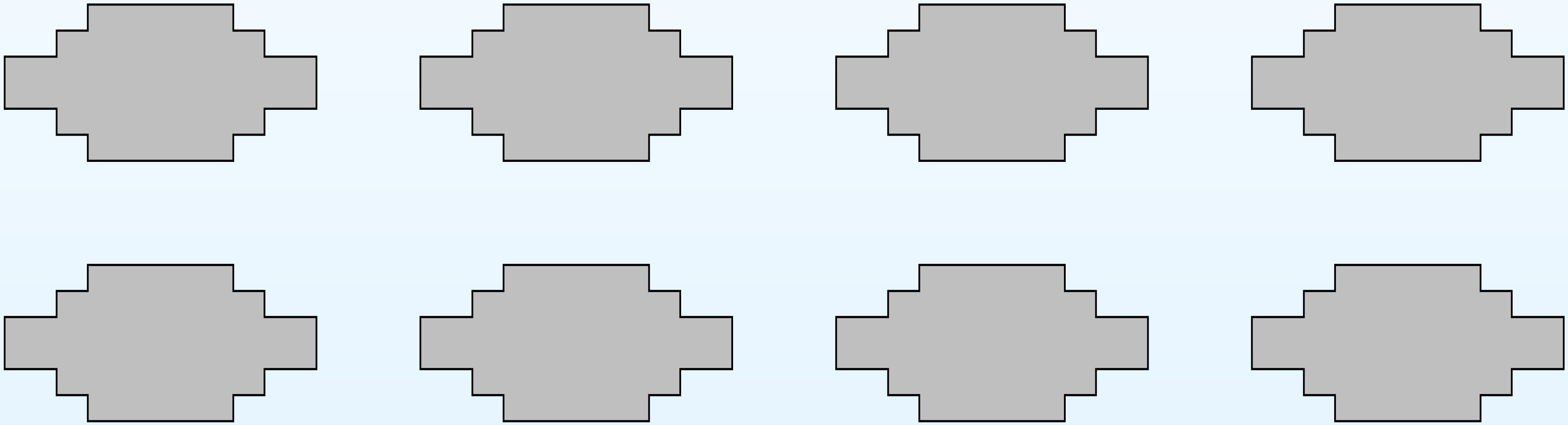
Changing the shape of the obstacle

Almost Old Theorem (V. Delecroix, A. Z., 2014). *Changing the shape of the obstacle we get a different diffusion rate. Say, for a symmetric obstacle with* $4m-4$ *angles* $3\pi/2$ *and* $4m$ *angles* $\pi/2$ *the diffusion rate is*

$$\frac{(2m)!!}{(2m+1)!!} \sim \frac{\sqrt{\pi}}{2\sqrt{m}} \quad \text{as } m \to \infty\,.$$



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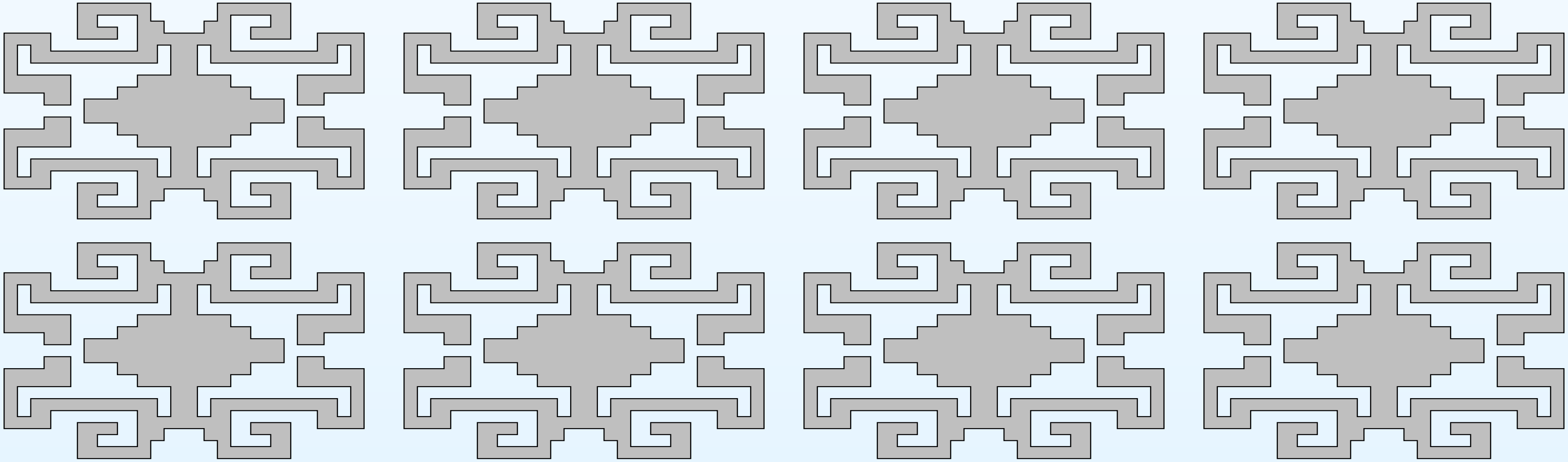

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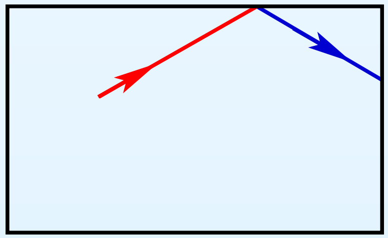
From a billiard to a surface foliation

Consider a rectangular billiard. Instead of reflecting the trajectory we can reflect the billiard table. The trajectory unfolds to a straight line. Folding back the copies of the billiard table we project this line to the original trajectory. At any moment the ball moves in one of four directions defining four types of copies of the billiard table. Copies of the same type are related by a parallel translation.



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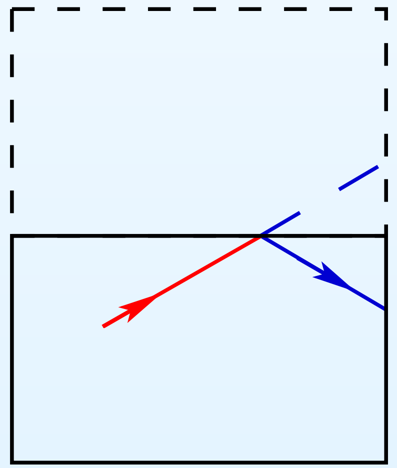

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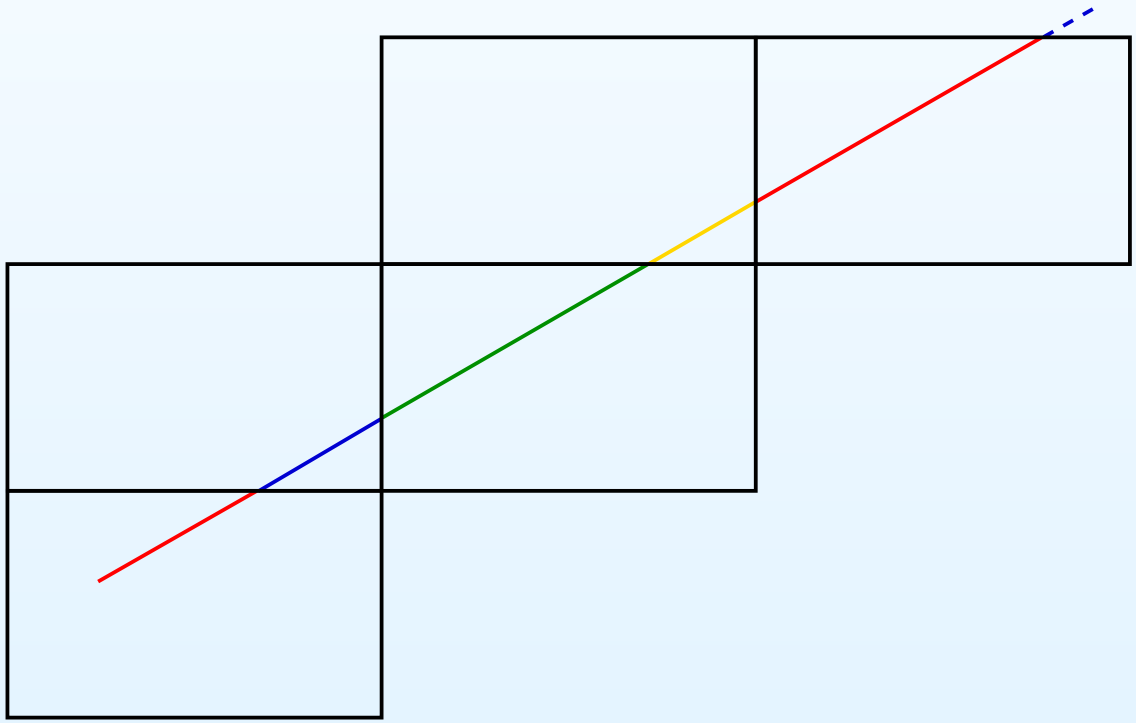

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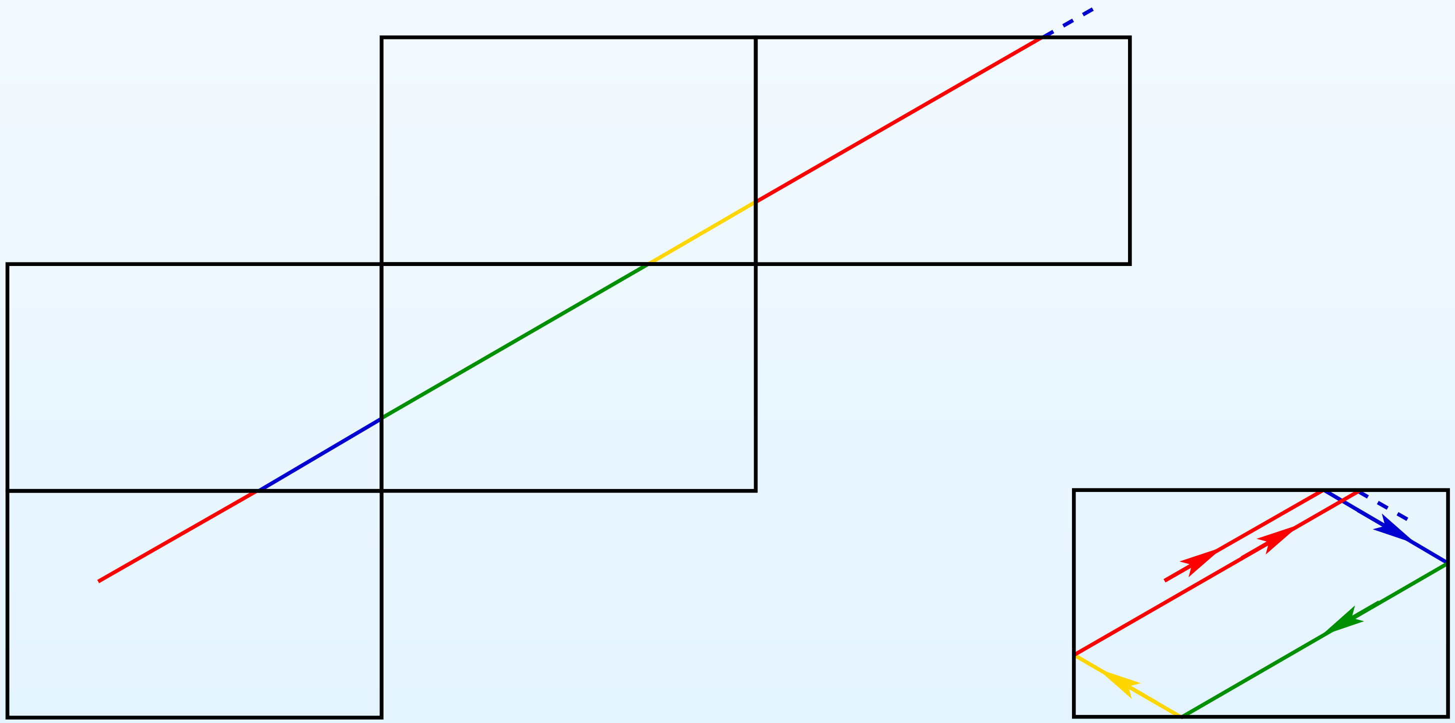

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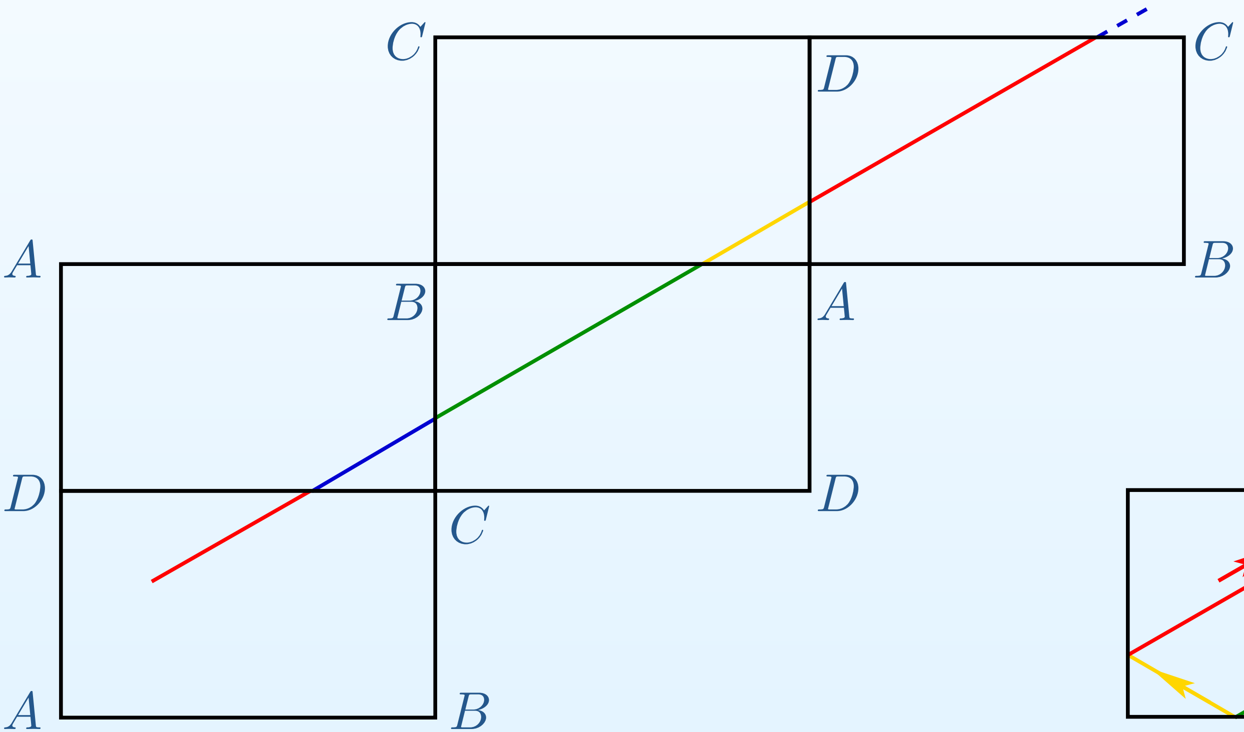

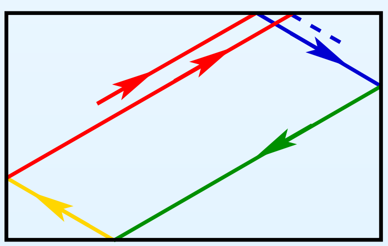

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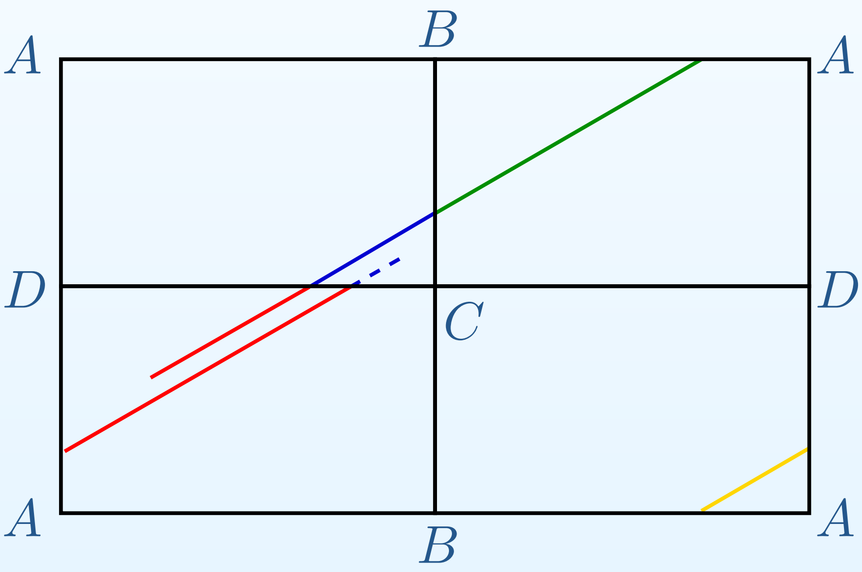
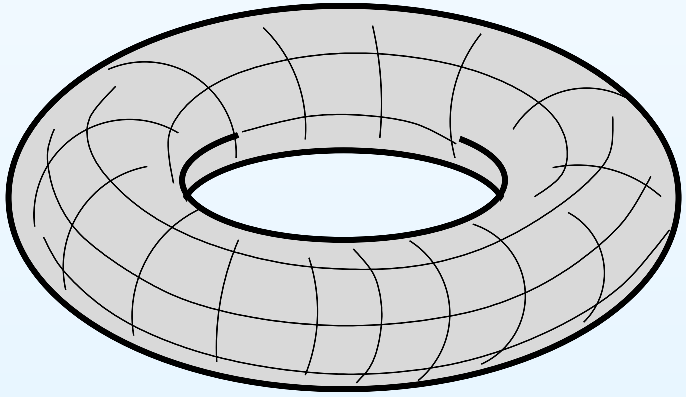
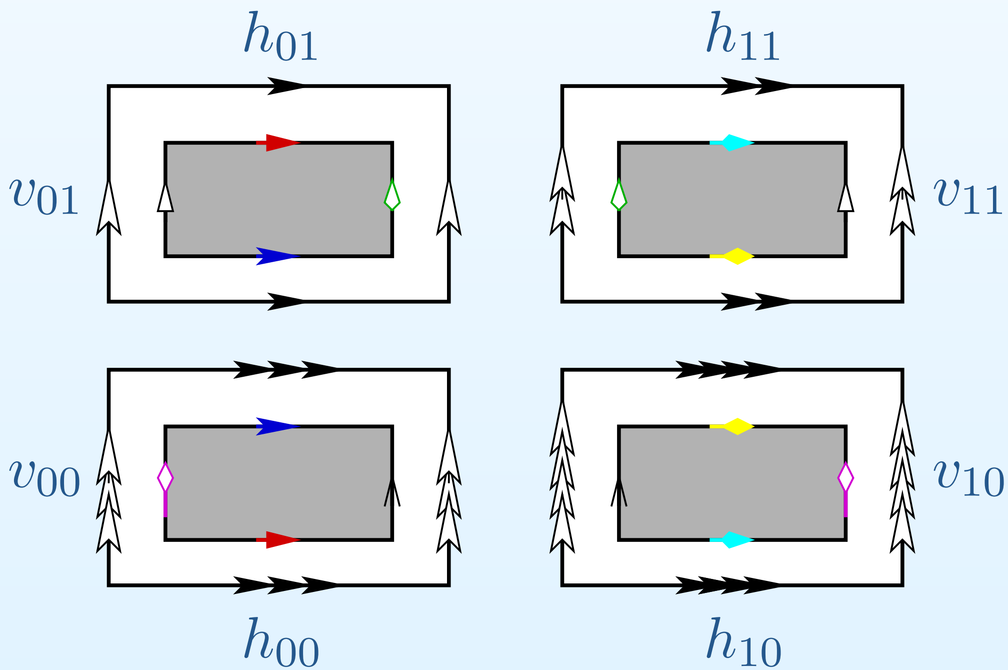




Identifying the equivalent patterns by a parallel translation we obtain a torus; the billiard trajectory unfolds to a "straight line" on the corresponding torus.

From the windtree billiard to a surface foliation

Similarly, taking four copies of our $\mathbb{Z}^2$-periodic windtree billiard we can unfold it to a foliation on a $\mathbb{Z}^2$-periodic surface. Taking a quotient over $\mathbb{Z}^2$ we get a compact surface endowed with a measured foliation. Vertical and horizontal displacement (and thus, the diffusion) of the billiard trajectories is described by the intersection numbers $c(t) \circ v$ and $c(t) \circ h$ of the cycle $c(t)$ obtained by closing up a long piece of leaf with the cycles $h = h_{00} + h_{10} - h_{01} - h_{11}$ and $v = v_{00} - v_{10} + v_{01} - v_{11}$.



0. Model problem: diffusion in a periodic billiard

1. Teichmüller dynamics (following ideas of B. Thurston)

● Diffeomorphisms of surfaces

● Pseudo-Anosov diffeomorphisms

● Space of lattices

● Moduli space of tori

● Very flat surface of genus 2

● Group action

● Magic of Masur—Veech Theorem

2. Asymptotic flag of an orientable measured foliation

3. State of the art

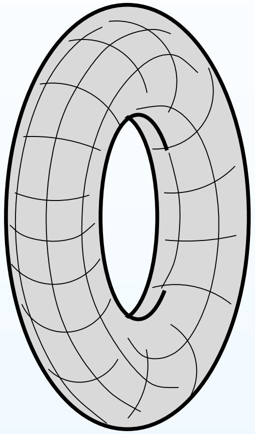
∞. What's next?

1. Teichmüller dynamics (following ideas of B. Thurston)

Diffeomorphisms of surfaces

Observation 1. *Surfaces can wrap around themselves.*

Cut a torus along a horizontal circle.



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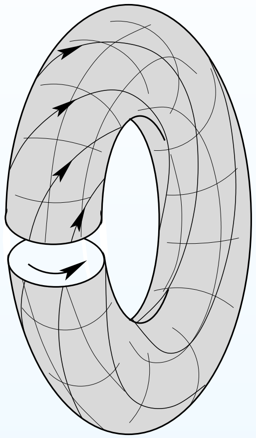
Twist de Dehn twists progressively horizontal circles up to a complete turn on the opposite boundary component of the cylinder and then identifies the components.



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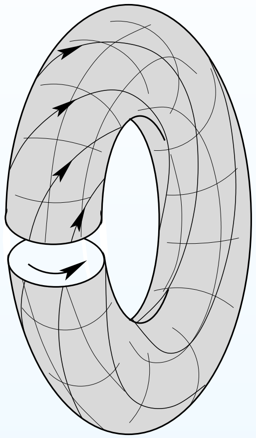

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\hat{f}_h} & \mathbb{R}^2 \\ \downarrow & & \downarrow \\ \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2 & \xrightarrow[f_h]{} & \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \end{array}$$

Dehn twist corresponds to the linear map $\hat{f}_h : \mathbb{R}^2 \to \mathbb{R}^2$ with the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

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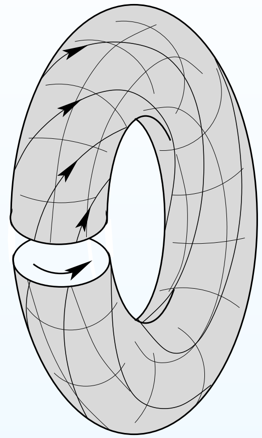

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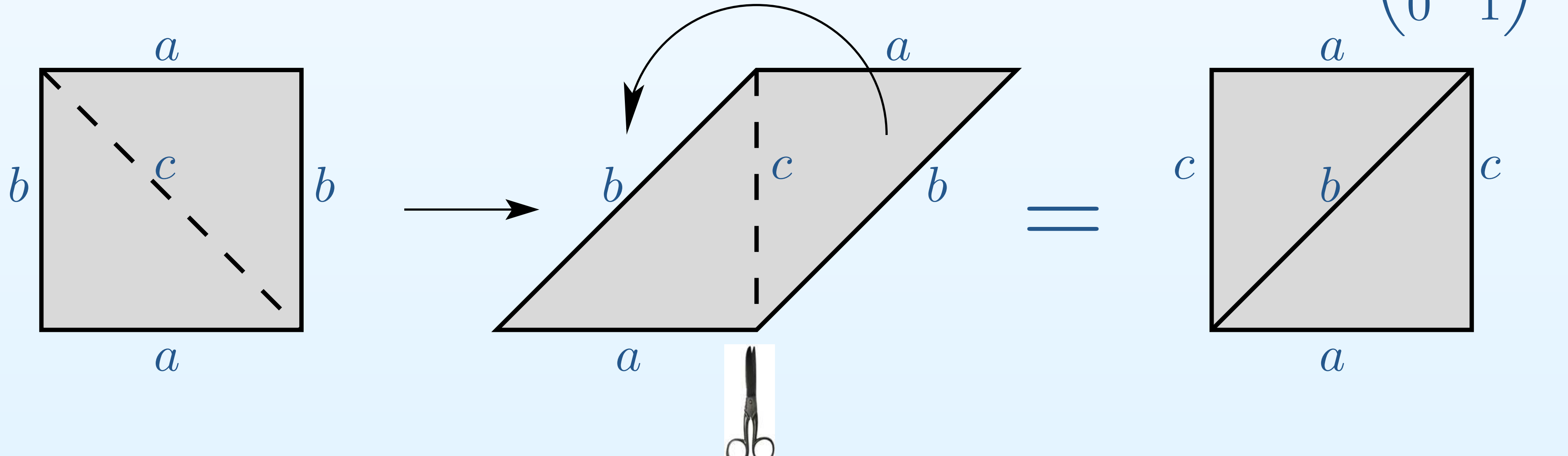
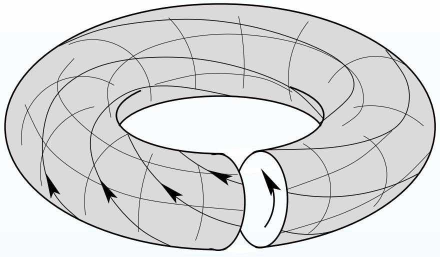
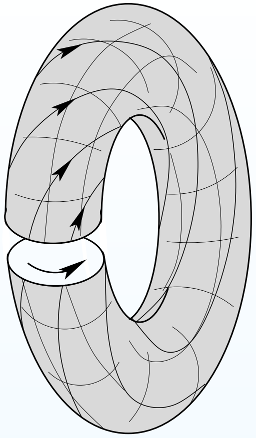


It maps the square pattern of the torus to a parallelogram pattern. Cutting and pasting appropriately we can transform the new pattern to the initial square one.

Pseudo-Anosov diffeomorphisms

Consider a composition
of two Dehn twists

$$g = f_v \circ f_h =$$  $\circ$ 

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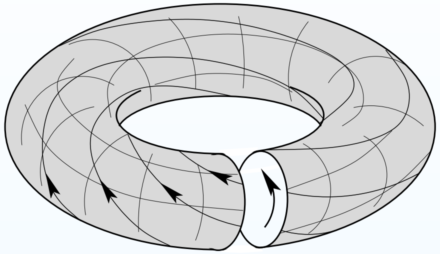
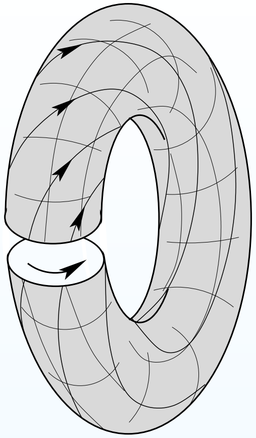
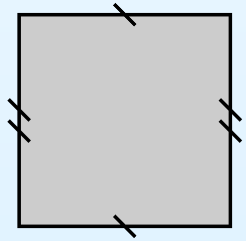
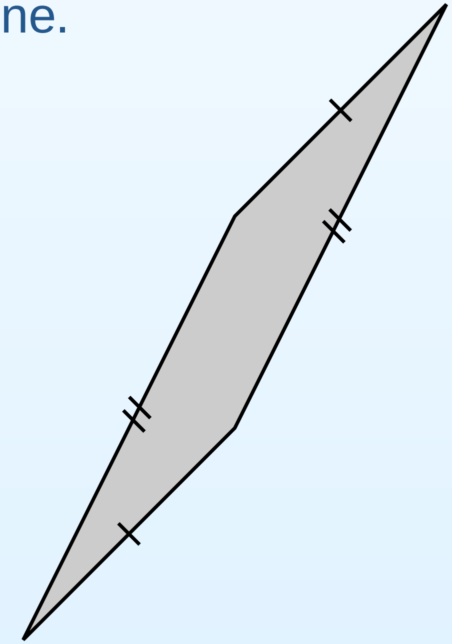
Consider a composition of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

It corresponds to the integer linear map $\hat{g} : \mathbb{R}^2 \to \mathbb{R}^2$ with matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Cutting and pasting appropriately the image parallelogram pattern we can check by hands that we can transform the new pattern to the initial square one.





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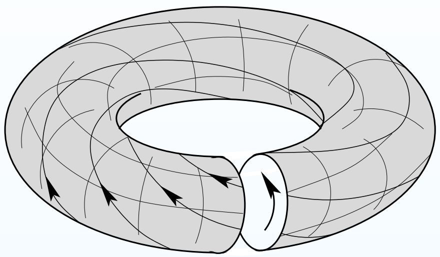
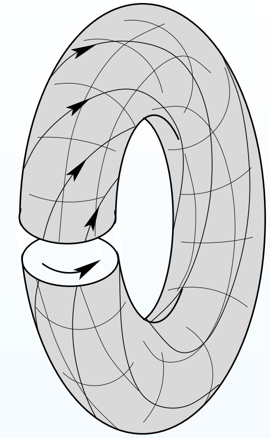
of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

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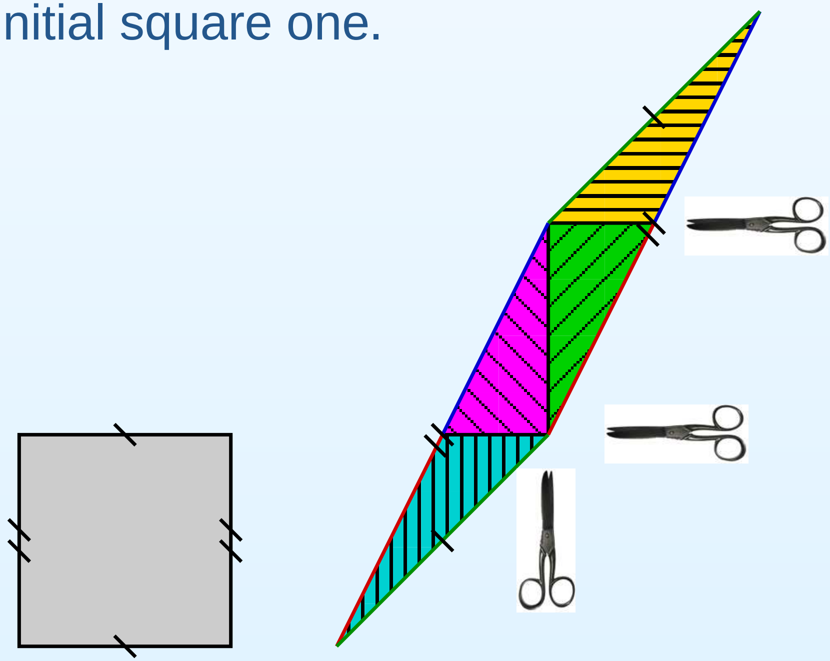

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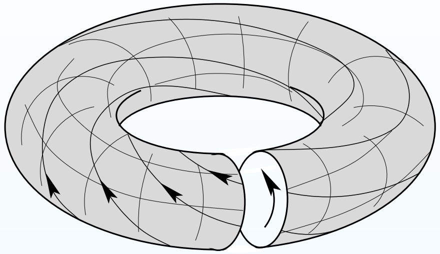
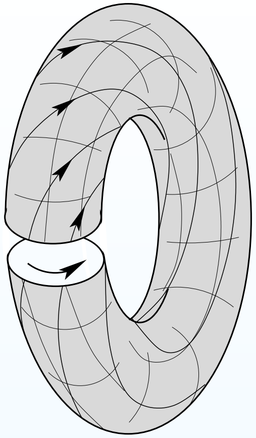
Consider a composition of two Dehn twists $g = f_v \circ f_h =$  $\circ$ 

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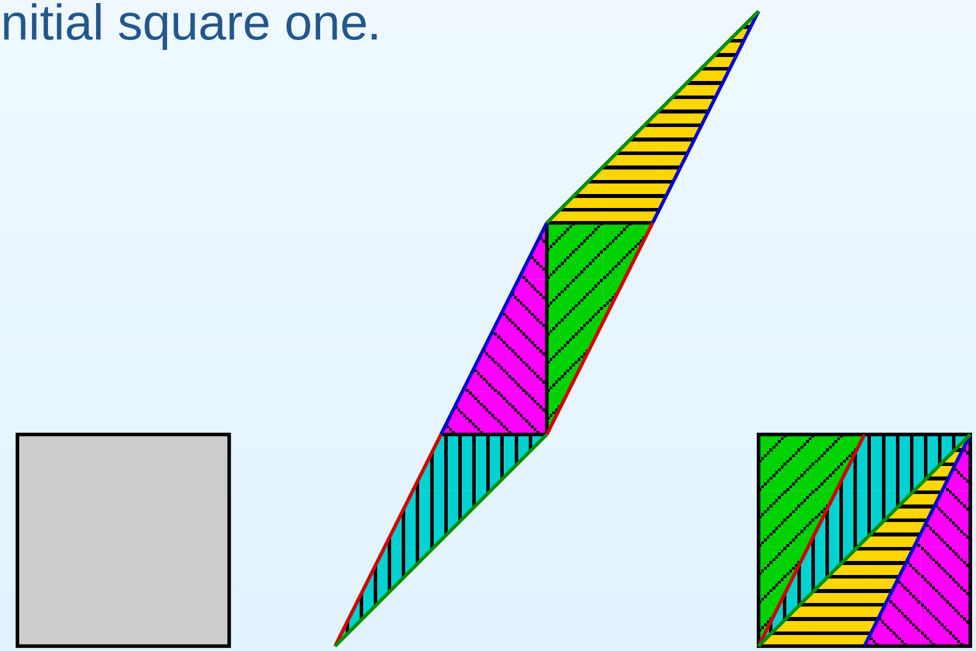

Pseudo-Anosov diffeomorphisms

Consider eigenvectors $\vec{v}_u$ and $\vec{v}_s$ of the linear transformation $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ with eigenvalues $\lambda = (3+\sqrt{5})/2 \approx 2.6$ and $1/\lambda = (3-\sqrt{5})/2 \approx 0.38$. Consider two transversal foliations on the original torus in directions $\vec{v}_u, \vec{v}_s$. We have just proved that expanding our torus $\mathbb{T}^2$ by factor λ in direction $\vec{v}_u$ and contracting it by the factor λ in direction $\vec{v}_s$ we get the original torus.

Definition. Surface automorphism homogeneously expanding in direction of one foliation and homogeneously contracting in direction of the transverse foliation is called a *pseudo-Anosov* diffeomorphism.

Consider a one-parameter family of flat tori obtained from the initial square torus by a continuous deformation expanding with a factor e^t in directions $\vec{v}_u$ and contracting with a factor e^t in direction $\vec{v}_s$. By construction such one-parameter family defines a closed curve in the space of flat tori: after the time $t_0 = \log \lambda_u$ it closes up and follows itself.

Observation 2. *Pseudo-Anosov diffeomorphisms define closed curves (actually, closed geodesics) in the moduli spaces of Riemann surfaces.*

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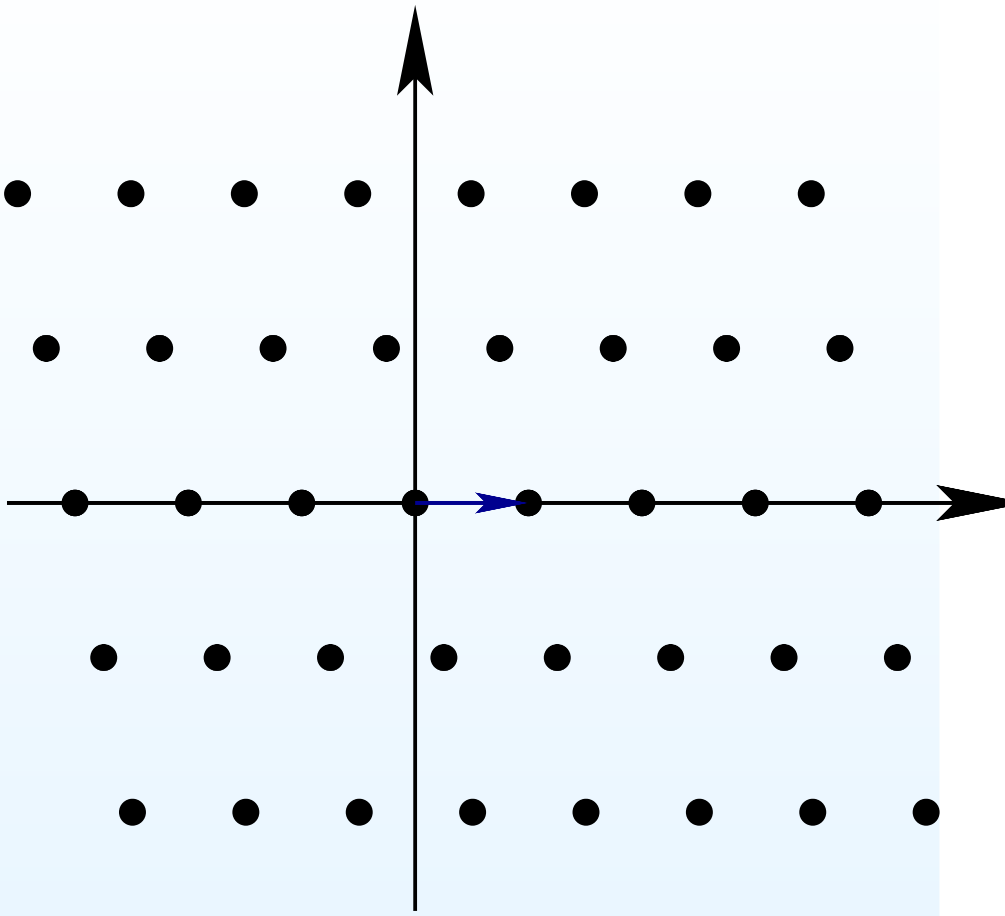
Space of lattices

- By a composition of homothety and rotation we can place the shortest vector of the lattice to the horizontal unit vector.



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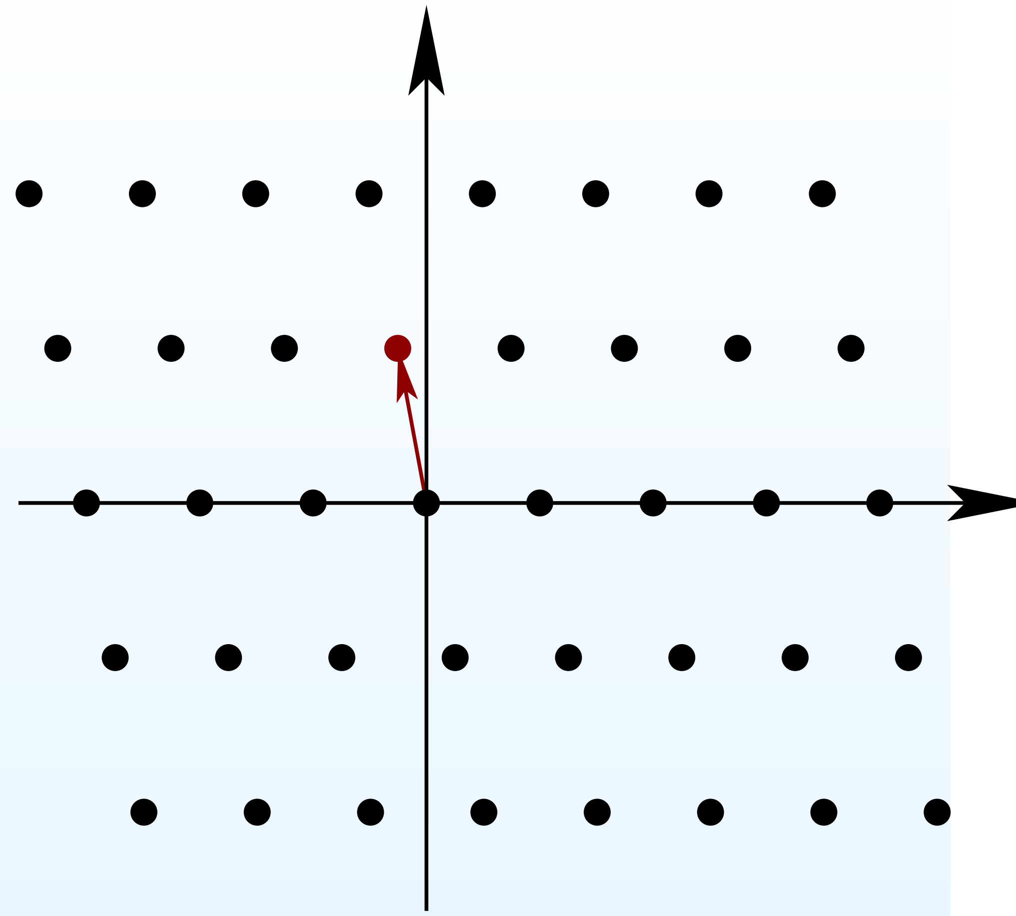

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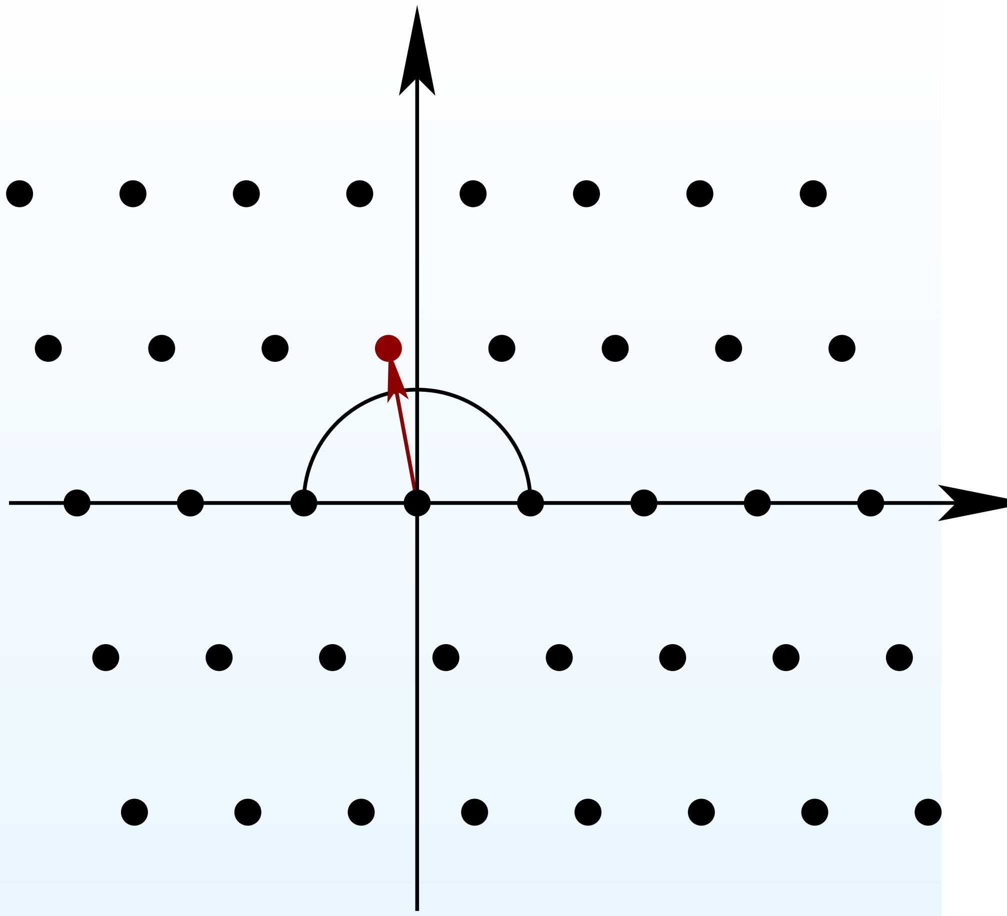

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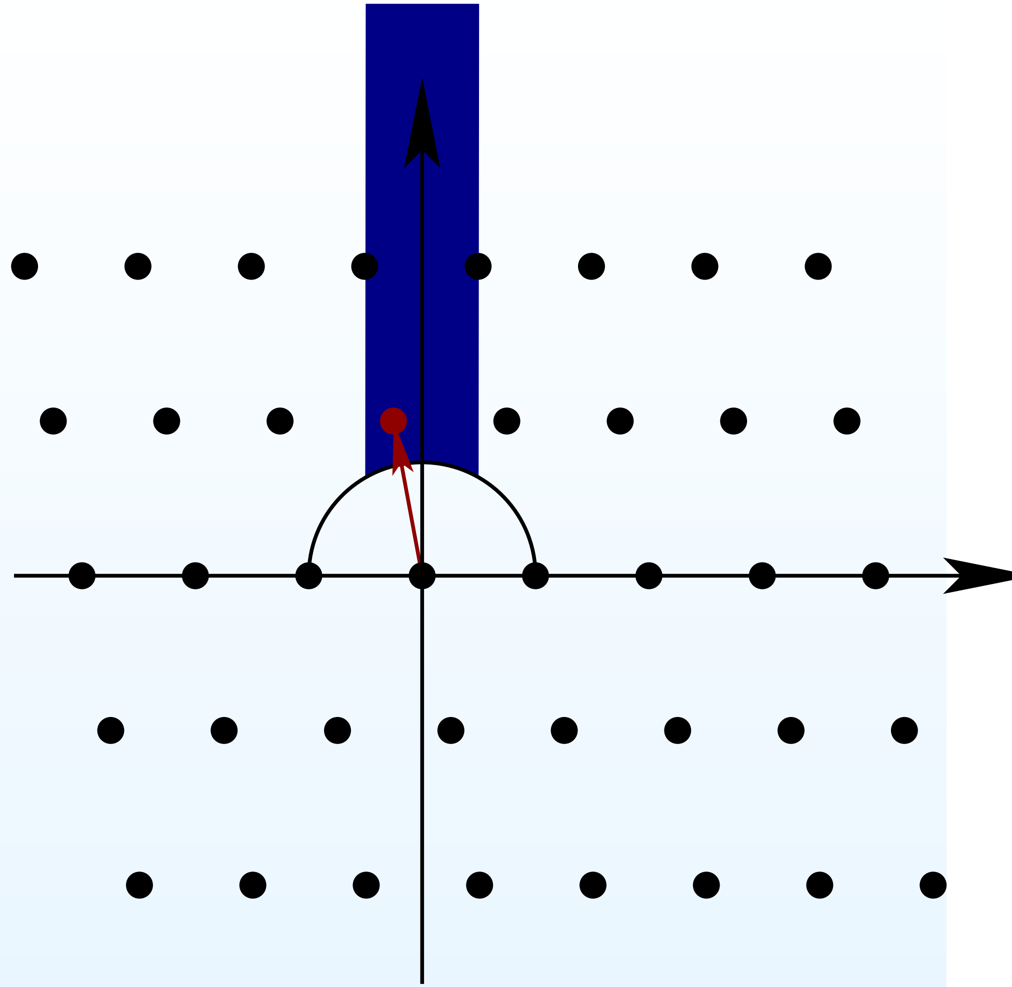

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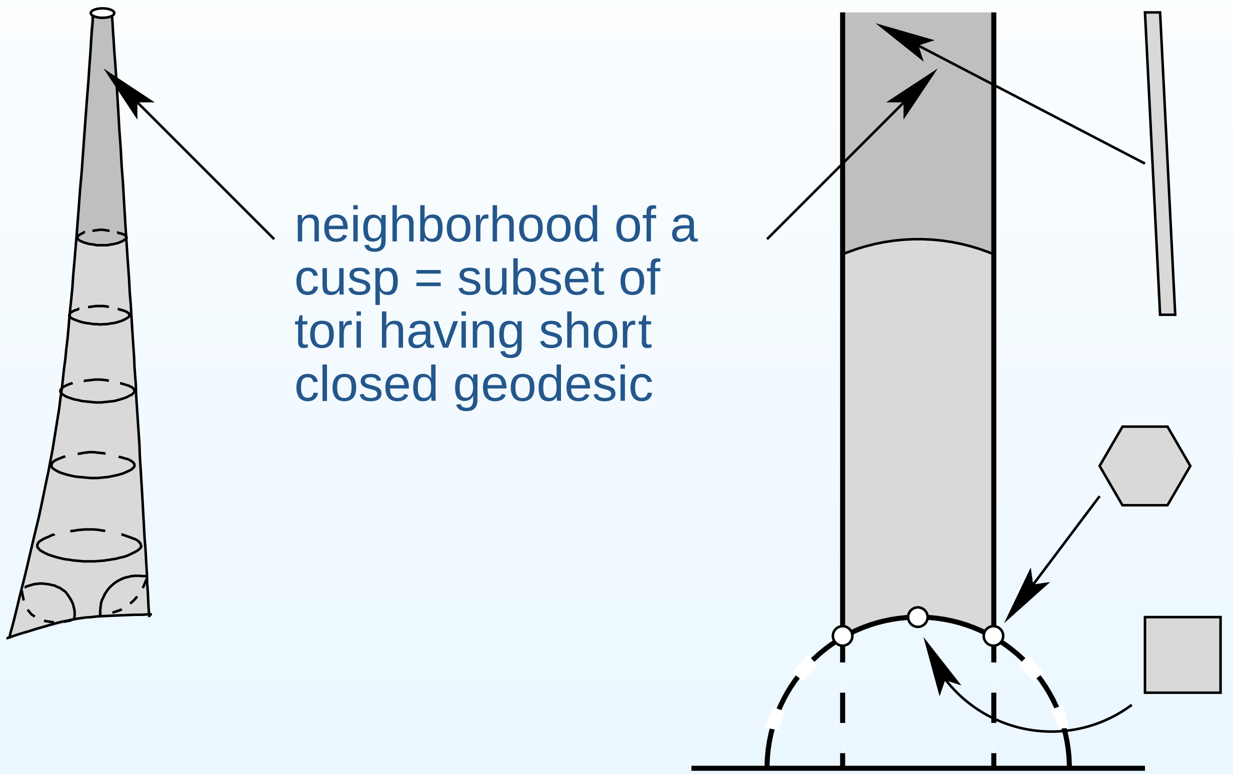
- Consider the lattice point closest to the origin and located in the upper half-plane.
- This point is located outside of the unit disc.
- It necessarily lives inside the strip $-1/2 \leq x \leq 1/2$.

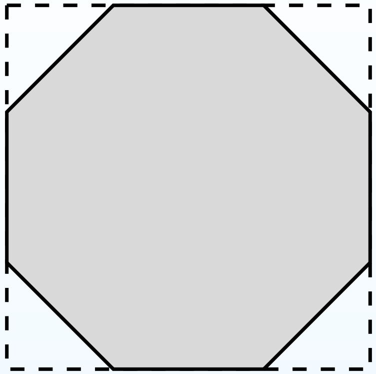
We get a fundamental domain in the space of lattices, or, in other words, in the moduli space of flat tori.

Moduli space of tori



The corresponding modular surface is not compact: flat tori representing points, which are close to the cusp, are almost degenerate: they have a very short closed geodesic. It also have orbifoldic points corresponding to tori with extra symmetries.

Very flat surface of genus 2



Identifying the opposite sides of a regular octagon we get a flat surface of genus two. All the vertices of the octagon are identified into a single conical singularity. We always consider such a flat surface endowed with a distinguished (say, vertical) direction. By construction, the holonomy of the flat metric is trivial. Thus, the vertical direction at a single point globally defines vertical and horizontal foliations.

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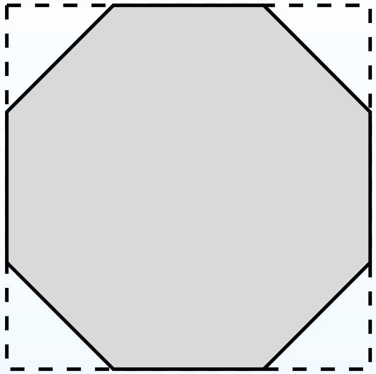

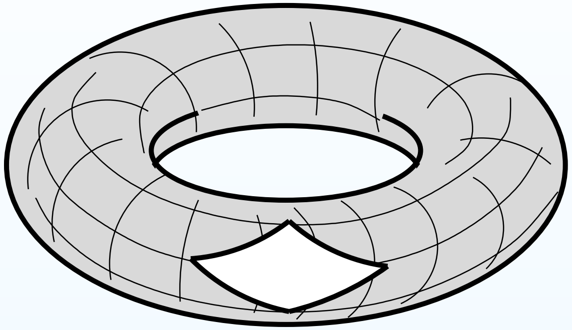

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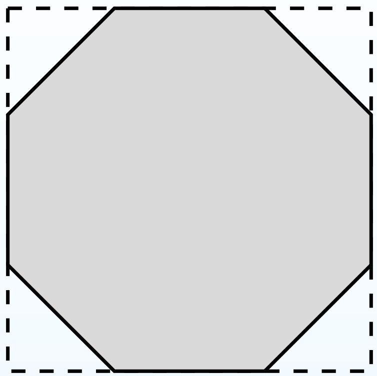

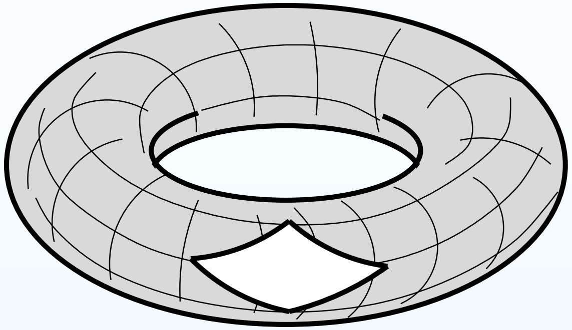

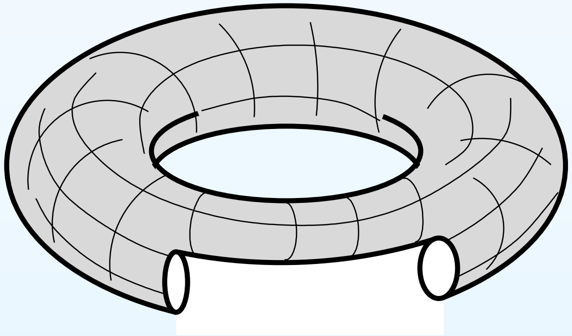

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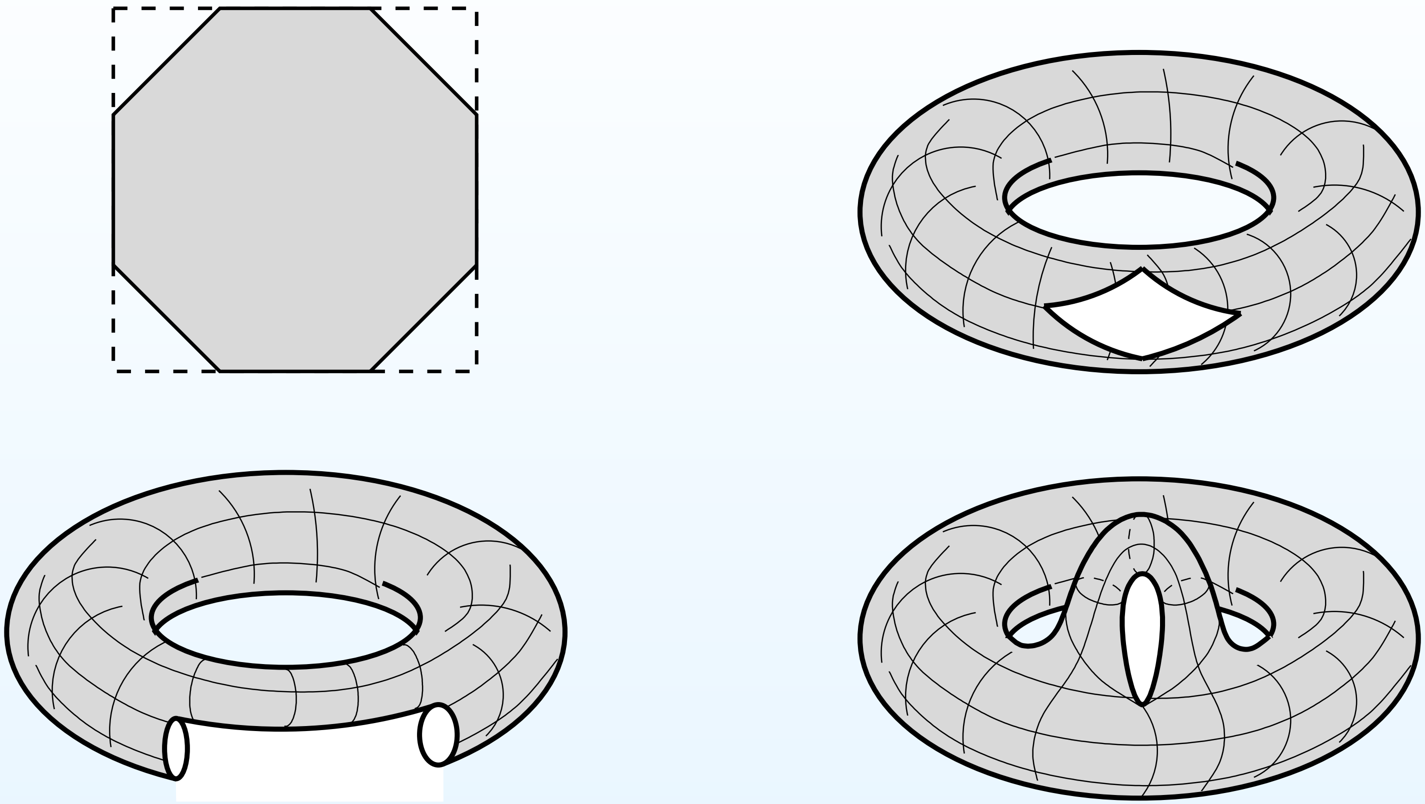

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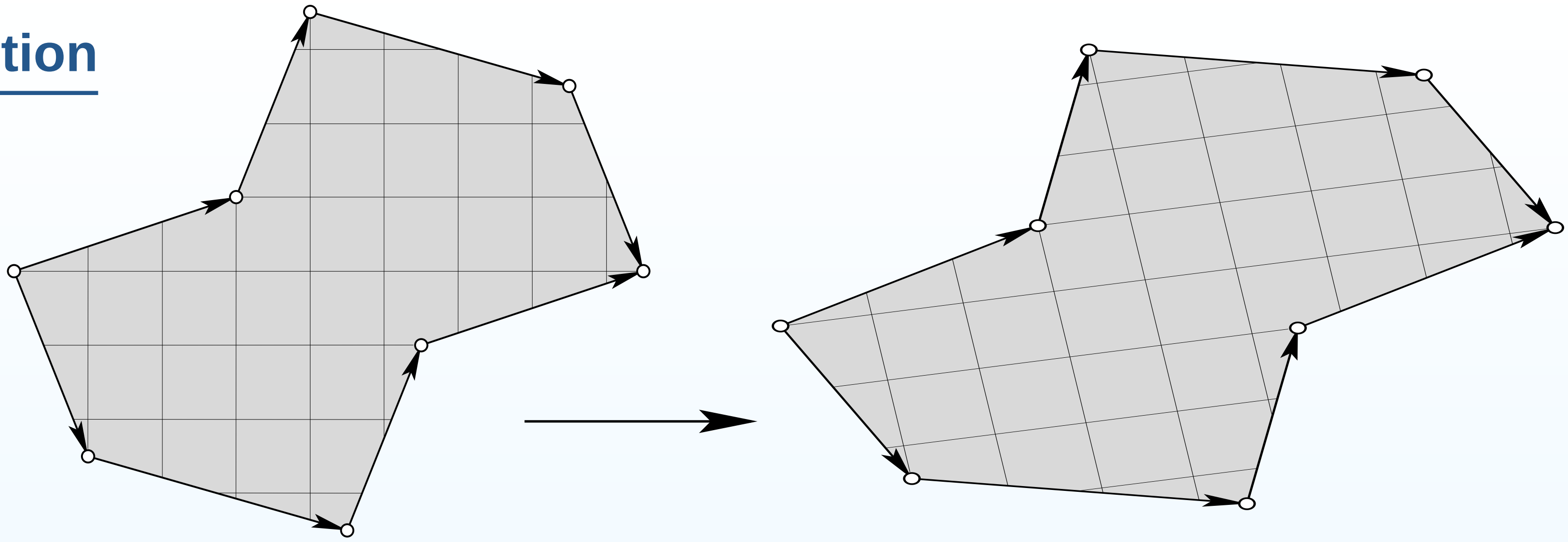
Group action



The group $\mathrm{SL}(2,\mathbb{R})$ acts on the each space $\mathcal{H}_1(d_1,\dots,d_n)$ of flat surfaces of unit area with conical singularities of prescribed cone angles $2\pi(d_i+1)$. This action preserves the natural measure on this space. The diagonal subgroup $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \subset \mathrm{SL}(2,\mathbb{R})$ induces a natural flow on $\mathcal{H}_1(d_1,\dots,d_n)$ called the *Teichmüller geodesic flow*.

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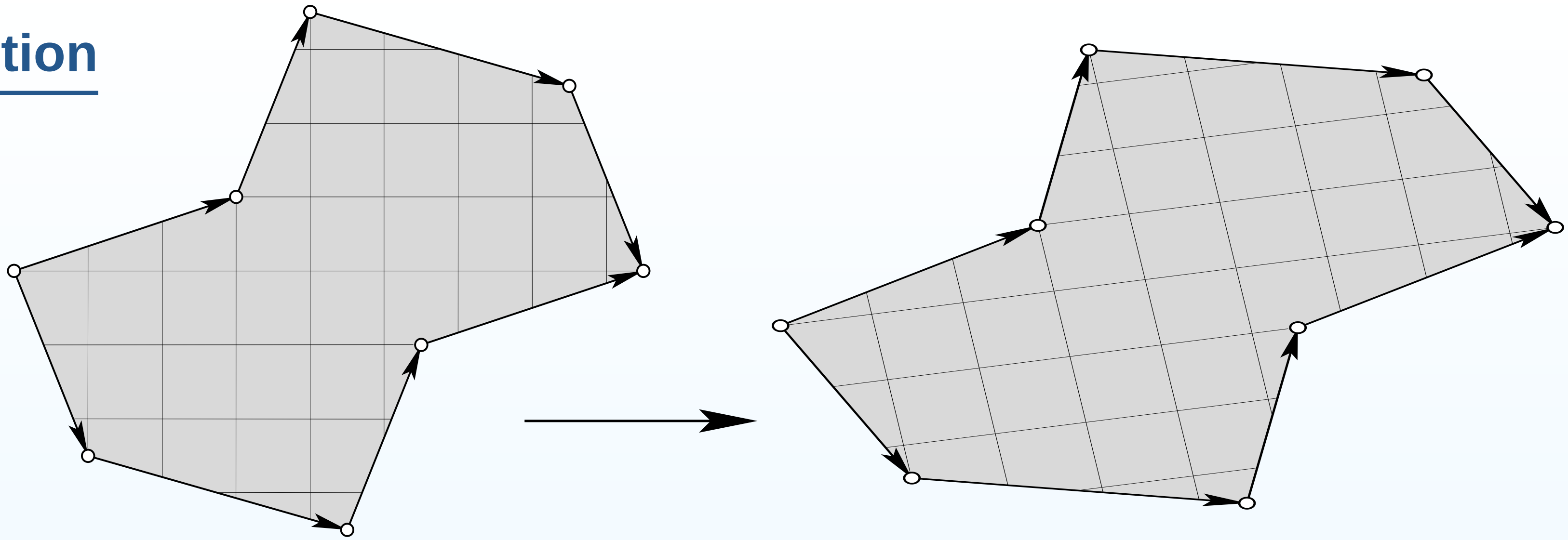

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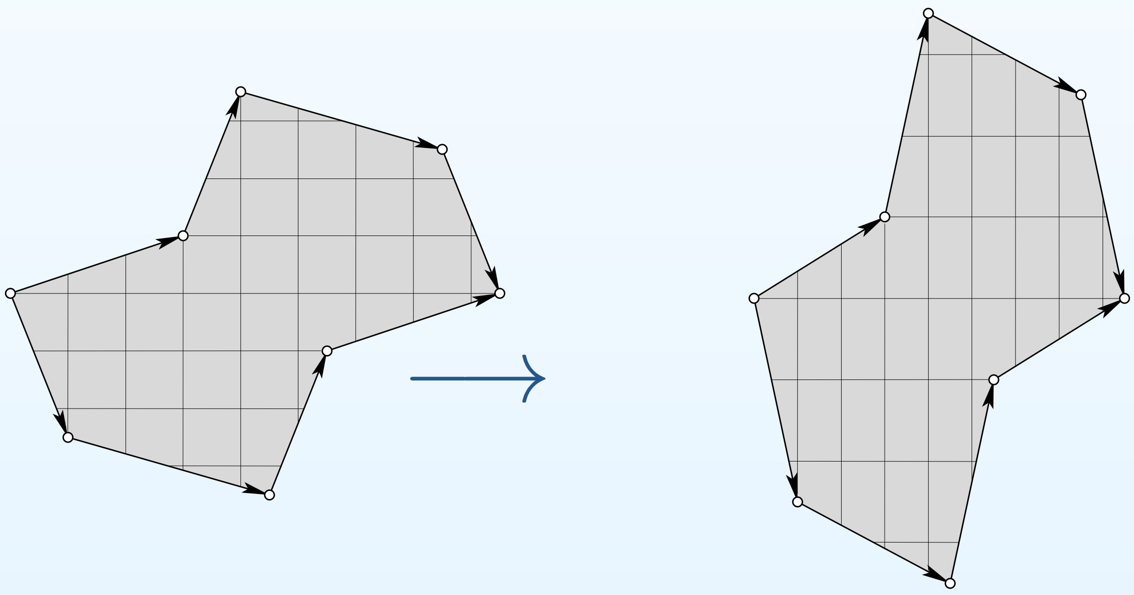

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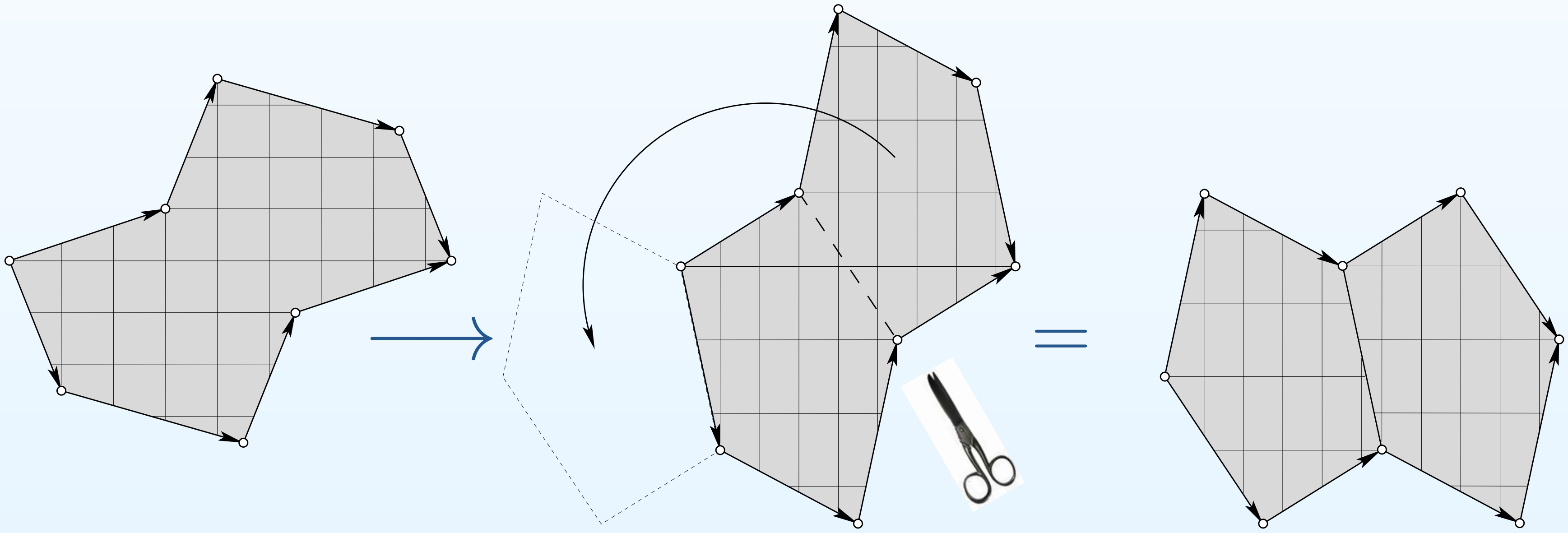
There is no paradox since we are allowed to cut-and-paste!



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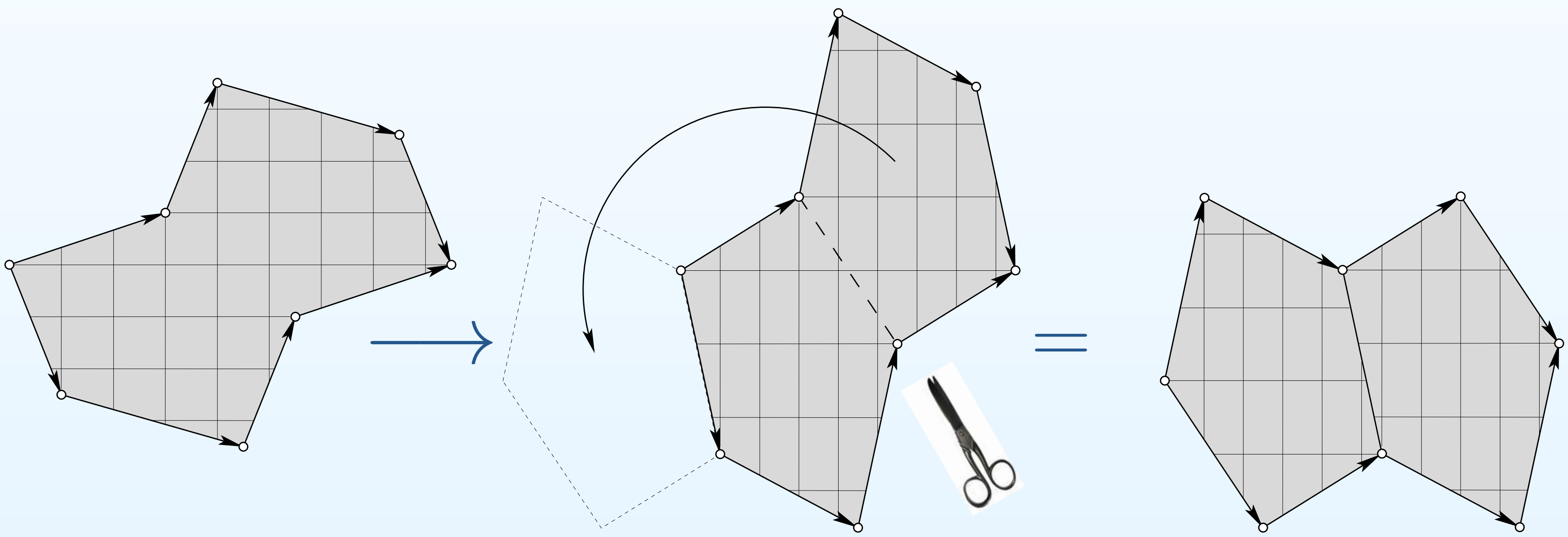


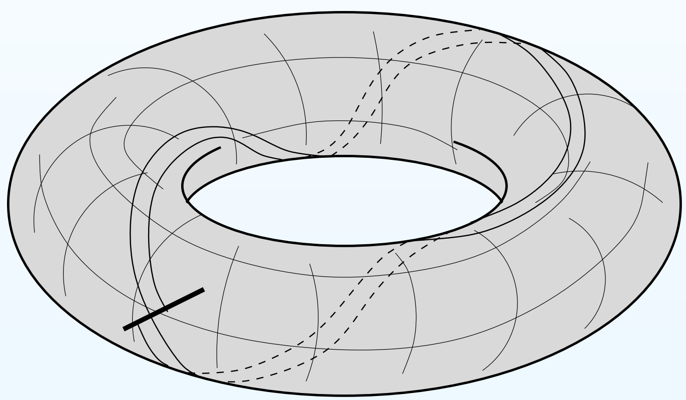
The first modification of the polygon changes the flat structure while the second one just changes the way in which we unwrap the flat surface.

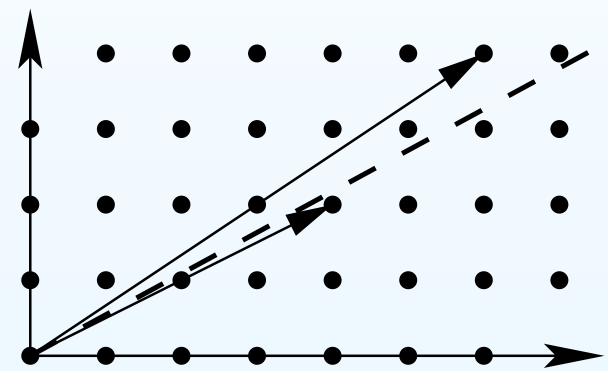
0. Model problem: diffusion in a periodic billiard
1. Teichmüller dynamics (following ideas of B. Thurston)
2. Asymptotic flag of an orientable measured foliation
● Asymptotic cycle
● First return cycles
● Renormalization
● Asymptotic flag: empirical description
● Multiplicative ergodic theorem
● Hodge bundle
● Other ingredients
3. State of the art
∞. What's next?

2. Asymptotic flag of an orientable measured foliation

Asymptotic cycle for a torus

Consider a leaf of a measured foliation on a surface. Choose a short transversal segment X. Each time when the leaf crosses X we join the crossing point with the point x_0 along X obtaining a closed loop. Consecutive return points $x_1, x_2, \ldots$ define a sequence of cycles $c_1, c_2, \ldots$.





The *asymptotic cycle* is defined as $\lim_{n\to\infty} \dfrac{c_n}{n} = c \in H_1(\mathbb{T}^2; \mathbb{R})$.

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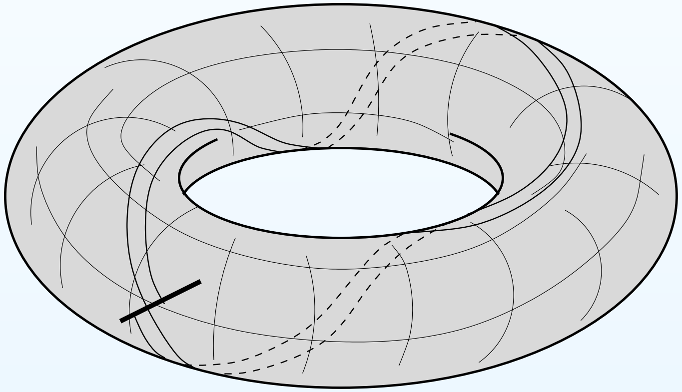

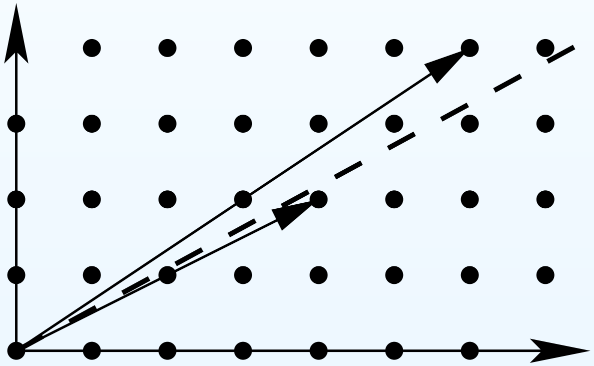

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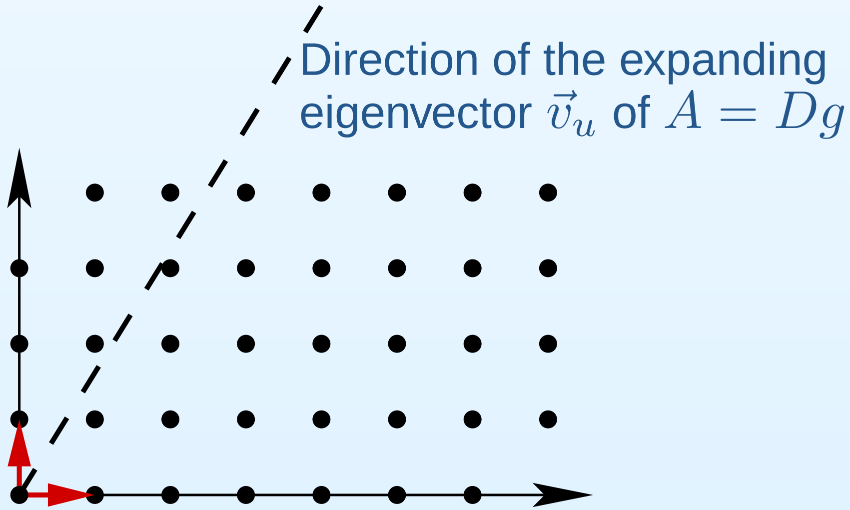
Asymptotic cycle in the pseudo-Anosov case

Consider a model case of the foliation in direction of the expanding eigenvector $\vec{v}_u$ of the Anosov map $g : \mathbb{T}^2 \to \mathbb{T}^2$ with $Dg = A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Take a closed curve γ and apply to it k iterations of g. The images $g_*^{(k)}(c)$ of the corresponding cycle $c = [\gamma]$ get almost collinear to the expanding eigenvector $\vec{v}_u$ of A, and the corresponding curve $g^{(k)}(\gamma)$ closely follows our foliation.

The first return cycles to a short subinterval exhibit exactly the same behavior by a simple reason that they are images of the first return cycles to a longer subinterval under a high iteration of g.



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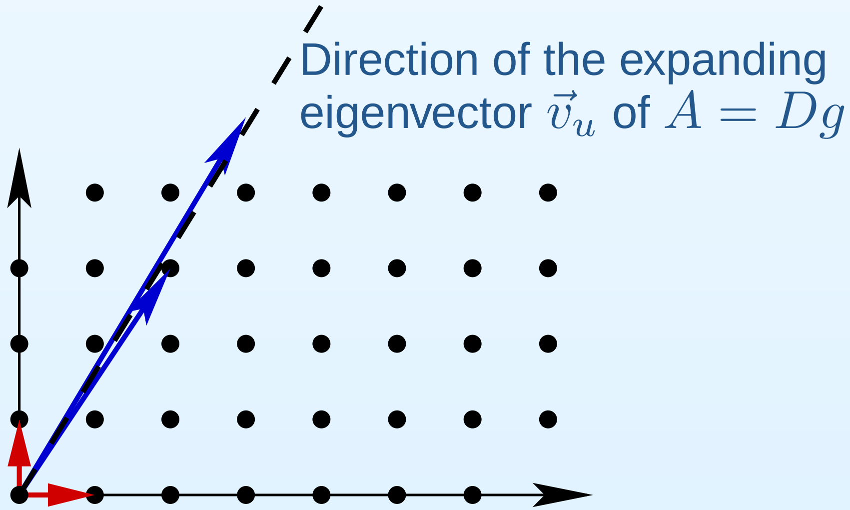

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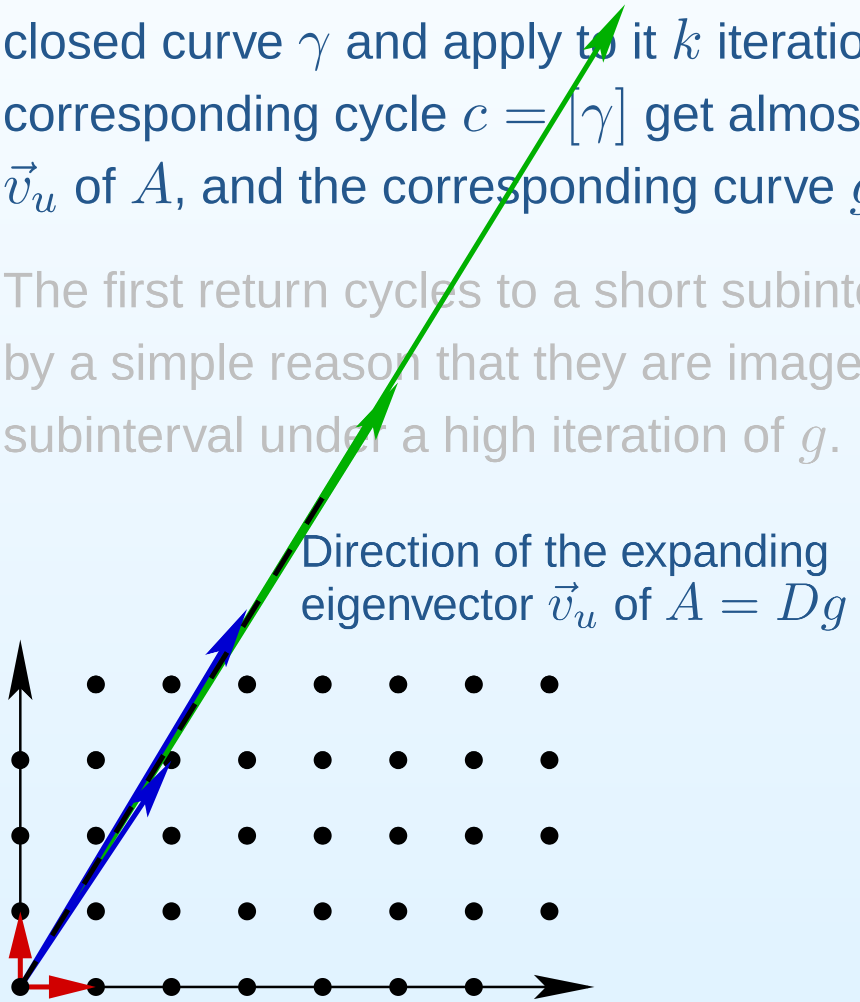

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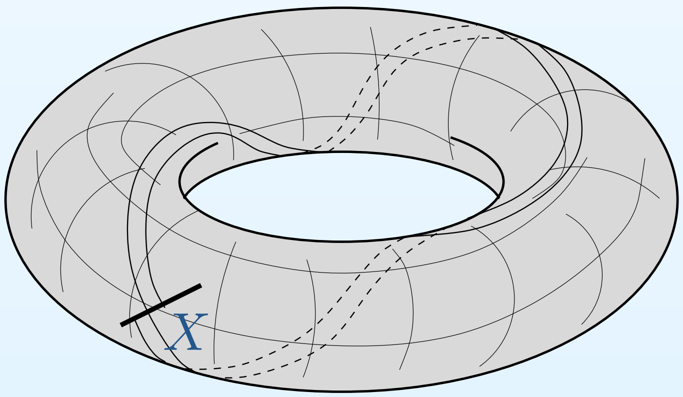
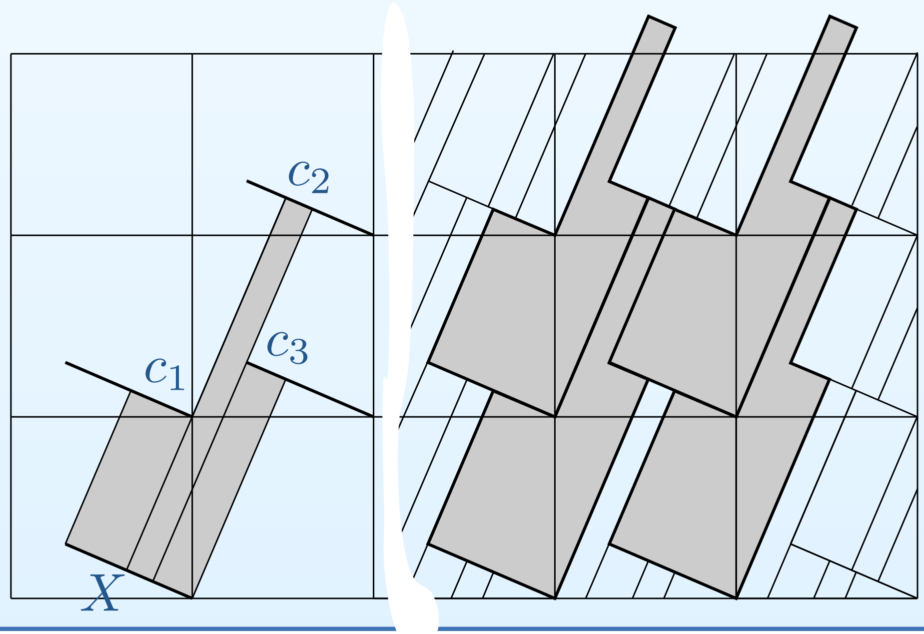
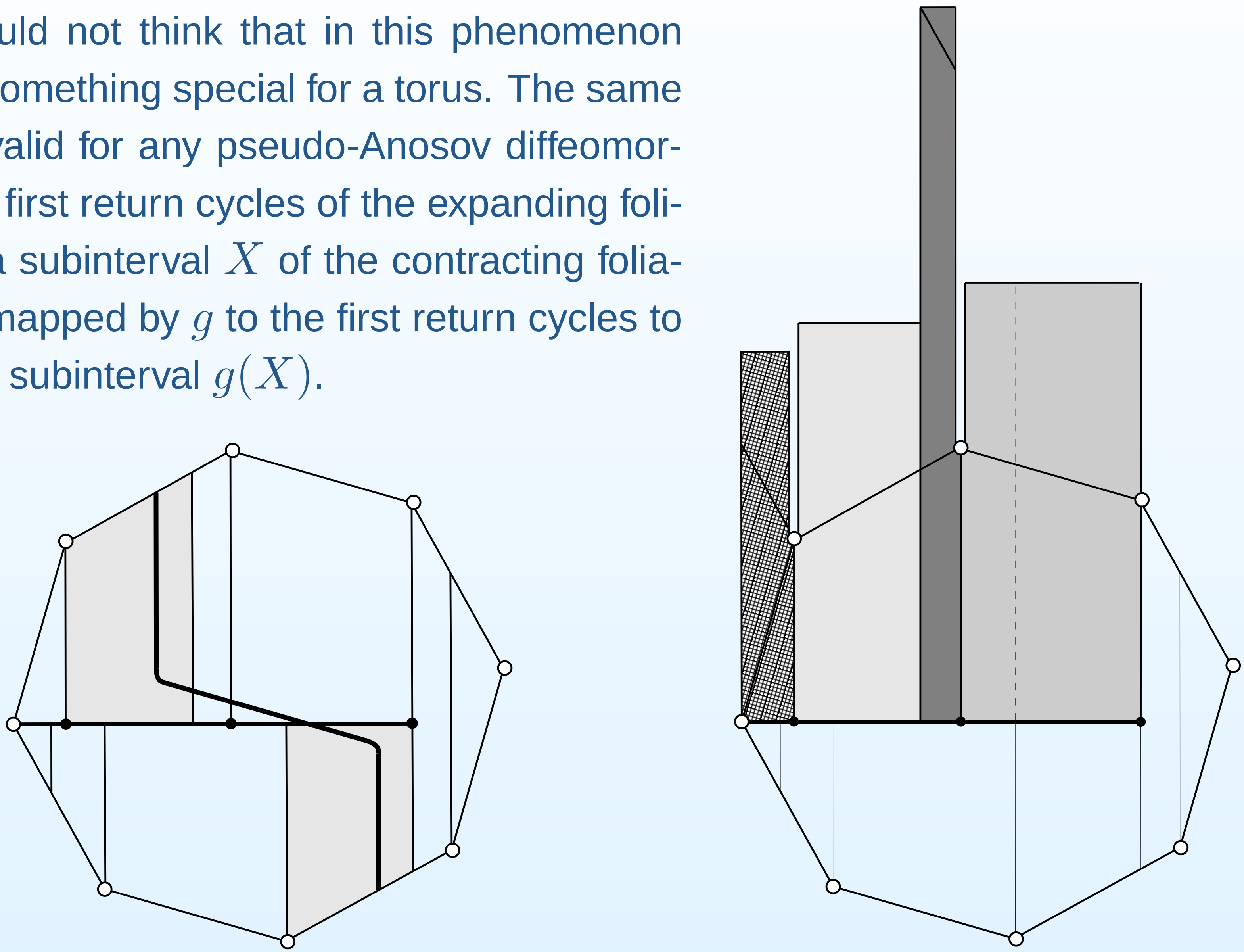




First return cycle $c_i(g(X))$ to $g(X)$ is $g_*(c_i(X))$

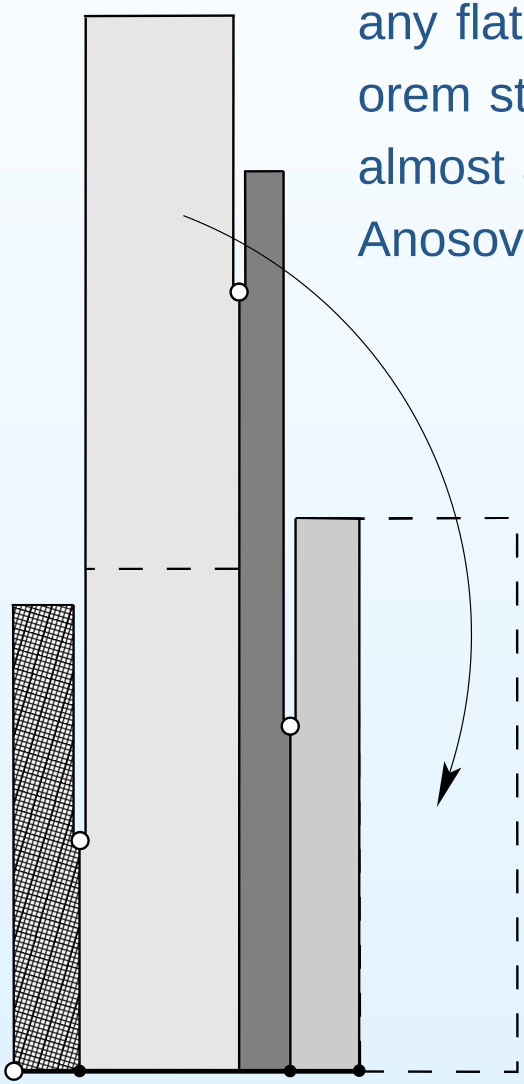
First return cycles

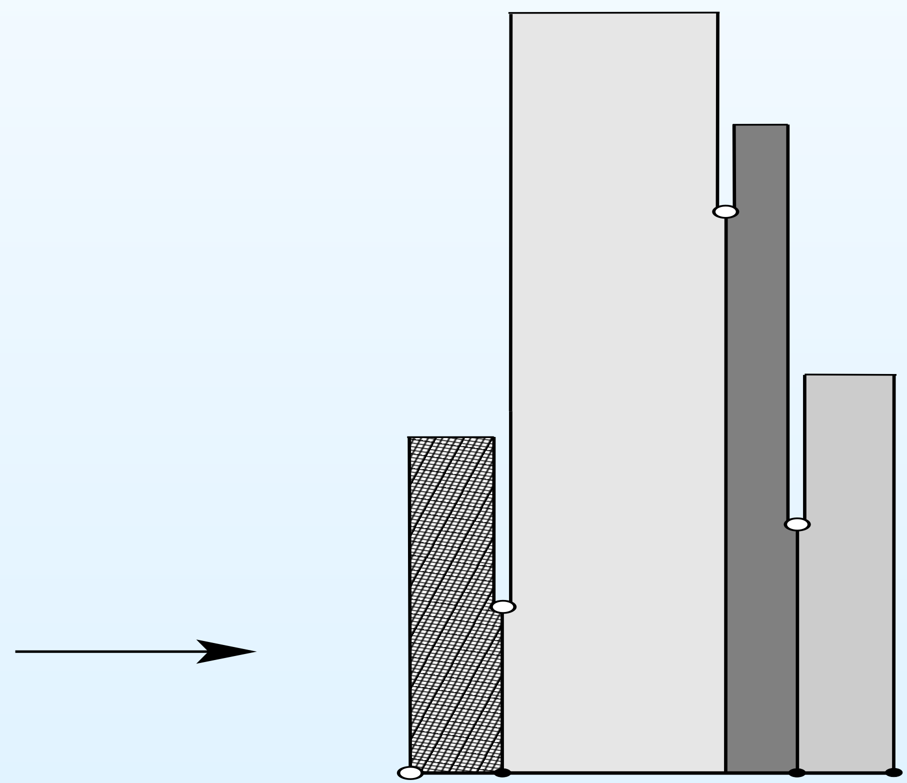
One should not think that in this phenomenon there is something special for a torus. The same story is valid for any pseudo-Anosov diffeomorphism g: first return cycles of the expanding foliation to a subinterval X of the contracting foliation are mapped by g to the first return cycles to a shorter subinterval $g(X)$.



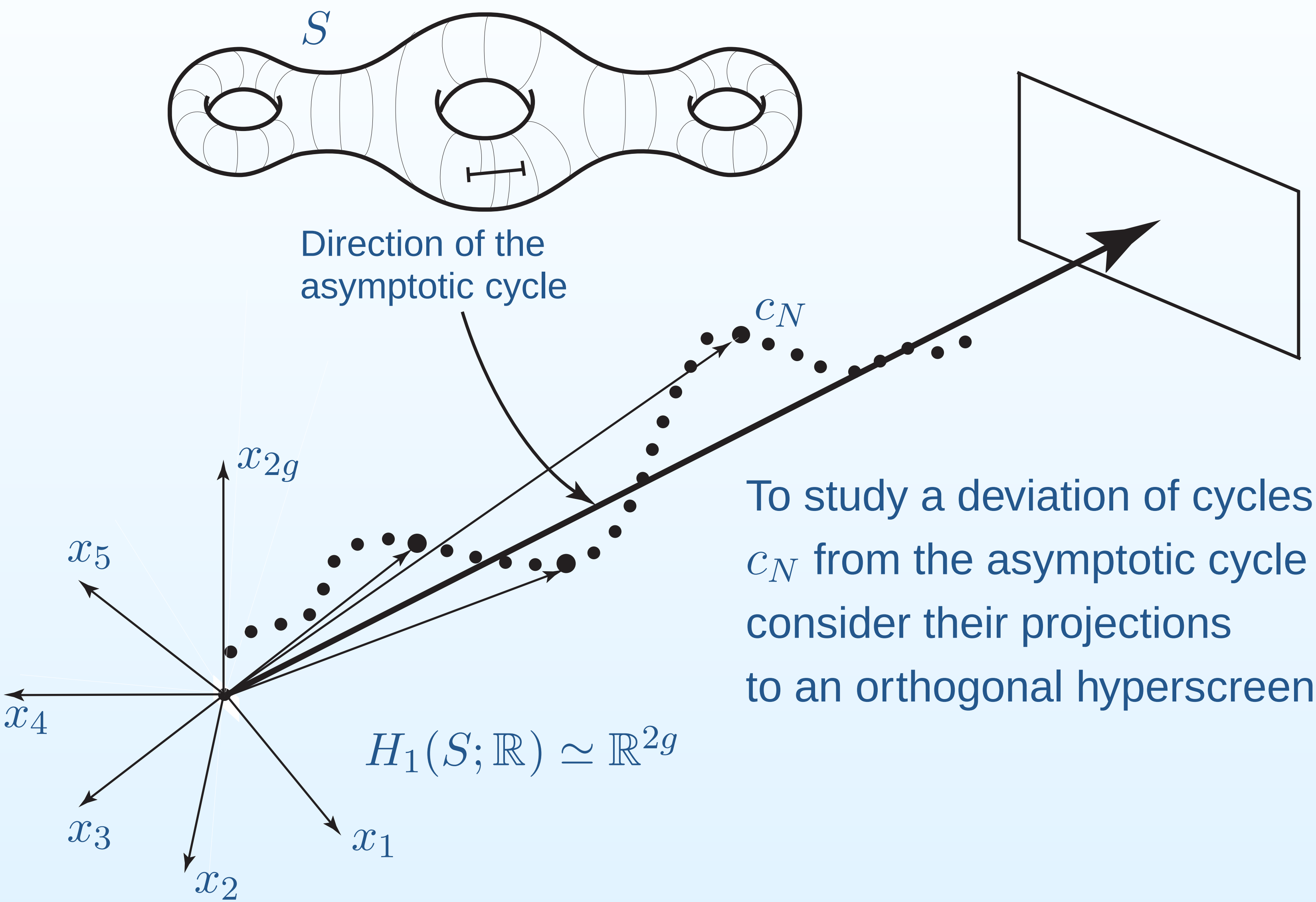
Idea of a renormalization

By the theorem of Masur and Veech, the homogeneous expansion-contraction in vertical-horizontal directions regularly brings almost any flat surface, basically, back to itself. Multiplicative ergodic theorem states that, in a sense, there a matrix (one and the same for almost all flat surfaces) which mimics the matrix of a fixed pseudo-Anosov diffeomorphism as if the Teichmüller flow would be periodic.





Asymptotic flag: empirical description



To study a deviation of cycles c_N from the asymptotic cycle consider their projections to an orthogonal hyperscreen

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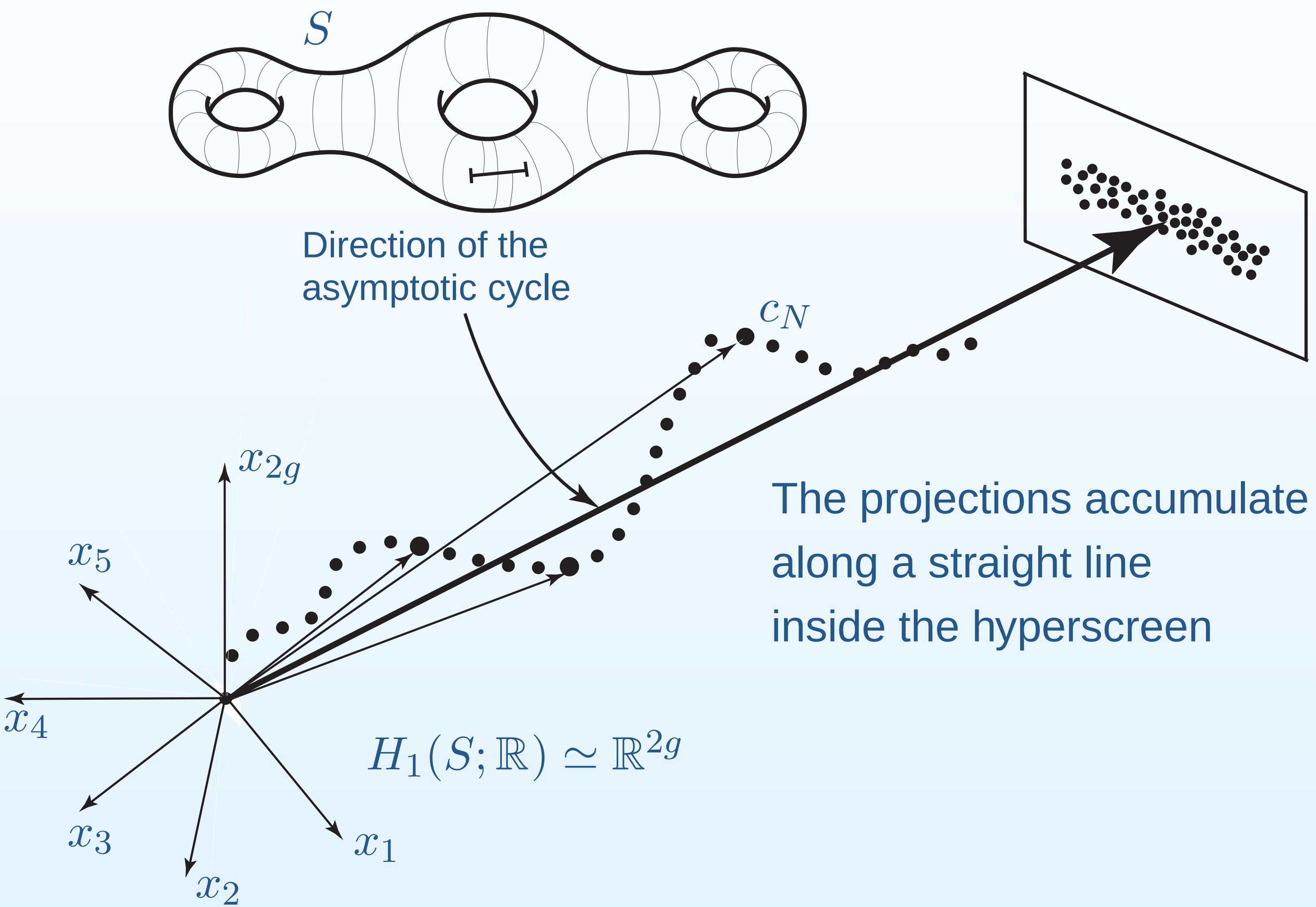

The projections accumulate along a straight line inside the hyperscreen

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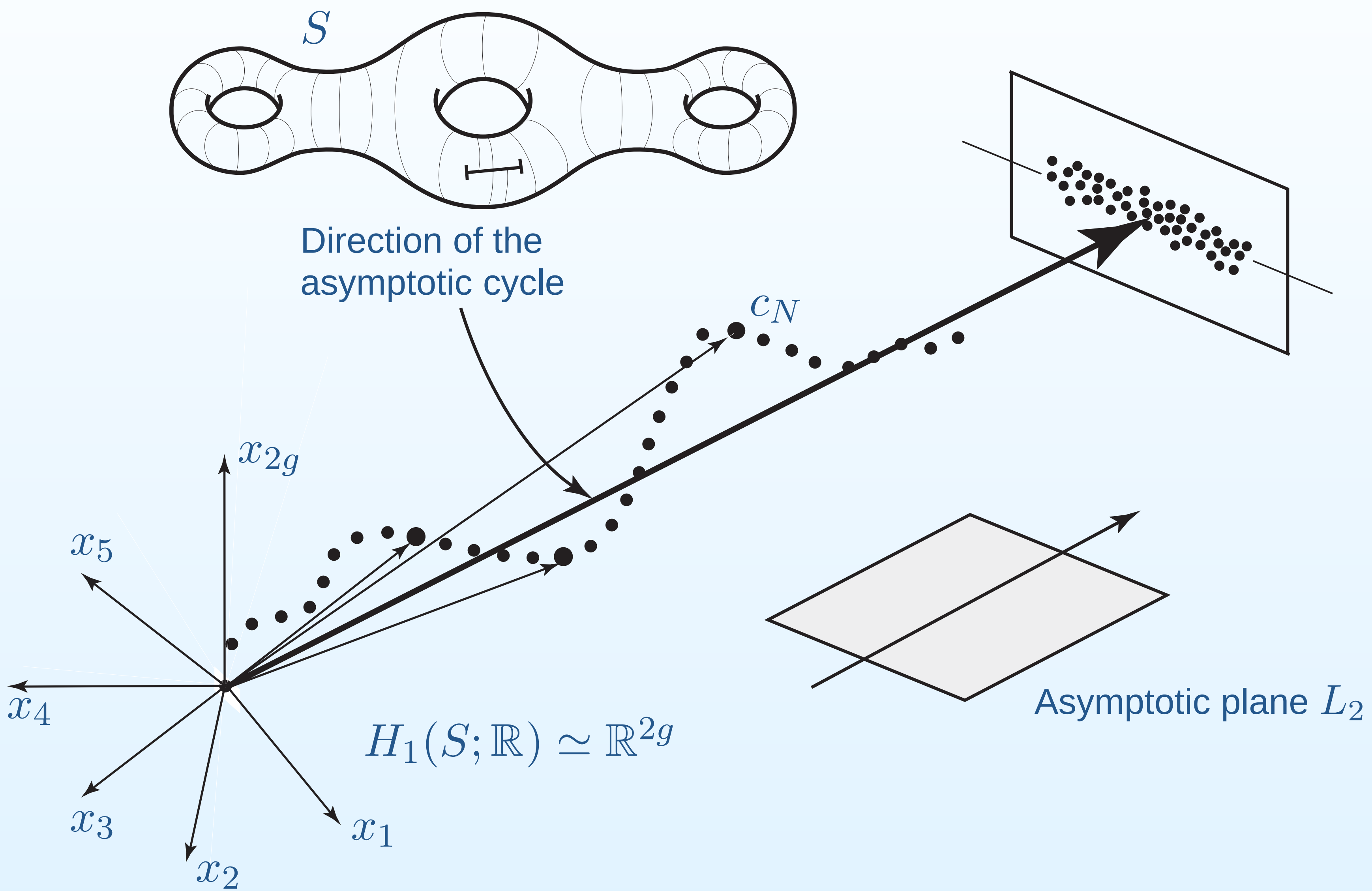

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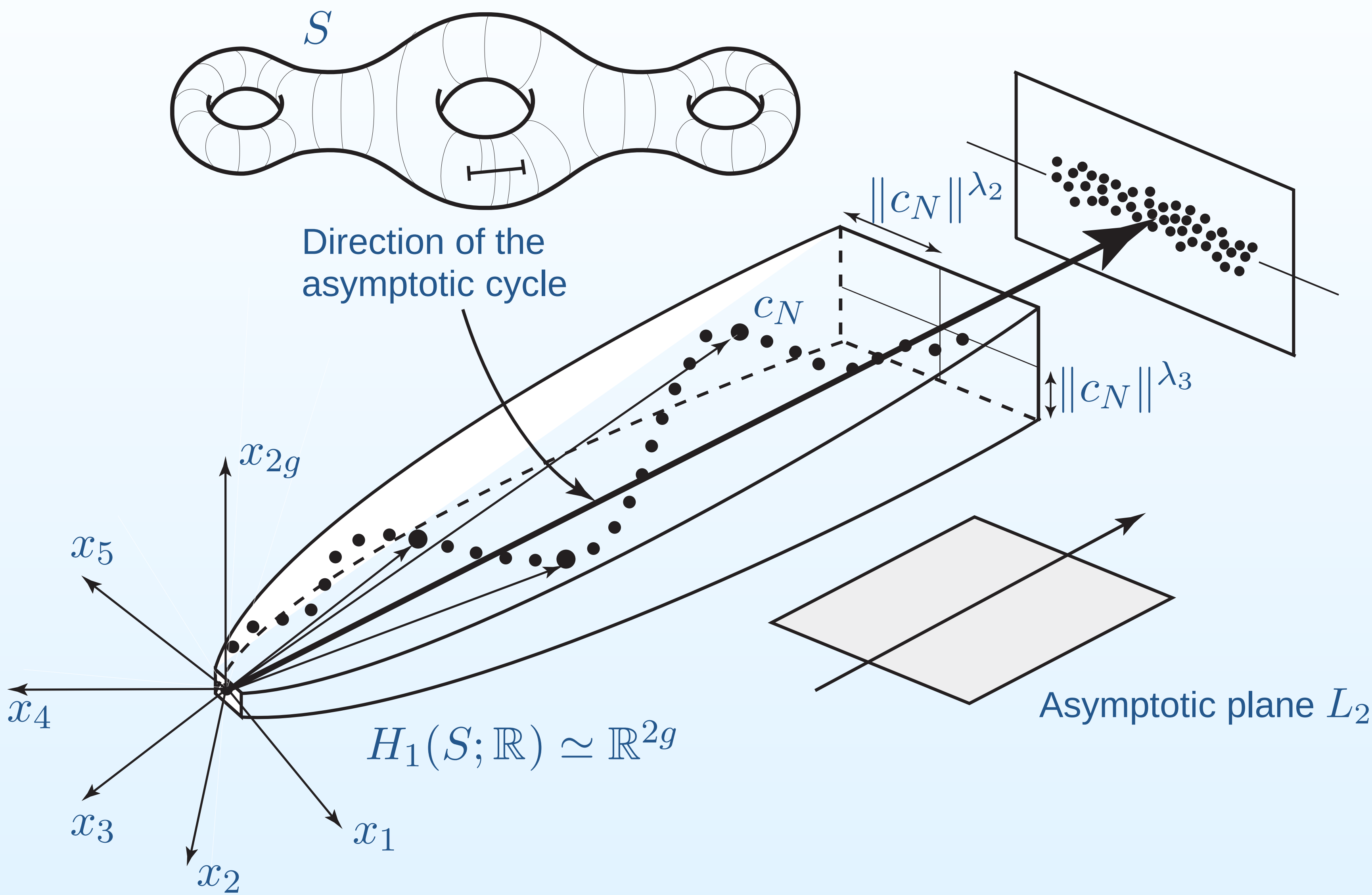

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Theorem (A. Z. , 1999) *For almost any surface S in any stratum $\mathcal{H}_1(d_1, \dots, d_n)$ there exists a flag of subspaces $L_1 \subset L_2 \subset \cdots \subset L_g \subset H_1(S;\mathbb{R})$ such that for any $j = 1, \dots, g-1$*

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and

$$\operatorname{dist}(c_N, L_g) \le const,$$

where the constant depends only on S and on the choice of the Euclidean structure in the homology space.

The numbers $1 = \lambda_1 > \lambda_2 > \cdots > \lambda_g$ are the top g Lyapunov exponents of the Hodge bundle along the Teichmüller geodesic flow on the corresponding connected component of the stratum $\mathcal{H}(d_1, \dots, d_n)$.

The strict inequalities $\lambda_g > 0$ and $\lambda_2 > \cdots > \lambda_g$, and, as a corollary, strict inclusions of the subspaces of the flag, are difficult theorems proved later by Forni (2002) and A. Avila–M. Viana (2007).

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Geometric interpretation of multiplicative ergodic theorem: spectrum of “mean monodromy”

Consider a vector bundle endowed with a flat connection over a manifold X^n. Having a flow on the base we can take a fiber of the vector bundle and transport it along a trajectory of the flow. When the trajectory comes close to the starting point we identify the fibers using the connection and we get a linear transformation $\mathcal{A}(x,1)$ of the fiber; the next time we get a matrix $\mathcal{A}(x,2)$, etc.

The multiplicative ergodic theorem says that when the flow is ergodic a *“matrix of mean monodromy”* along the flow

$$A_{mean} := \lim_{N\to\infty} \left(\mathcal{A}^*(x,N)\cdot\mathcal{A}(x,N)\right)^{\frac{1}{2N}}$$

is well-defined and constant for almost every starting point.

Lyapunov exponents correspond to logarithms of eigenvalues of this “matrix of mean monodromy”.

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is well-defined and constant for almost every starting point.

Lyapunov exponents correspond to logarithms of eigenvalues of this "matrix of mean monodromy".

Hodge bundle and Gauss–Manin connection

Consider a natural vector bundle over the stratum with a fiber $H^1(S;\mathbb{R})$ over a "point" (S,ω), called the *Hodge bundle*. It carries a canonical flat connection called *Gauss—Manin connection*: we have a lattice $H^1(S;\mathbb{Z})$ in each fiber, which tells us how we can locally identify the fibers. Thus, Teichmüller flow on $\mathcal{H}_1(d_1,\ldots,d_n)$ defines a multiplicative cocycle acting on fibers of this bundle.

The monodromy matrices of this cocycle are symplectic which implies that the Lyapunov exponents are symmetric:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq -\lambda_g \geq \cdots \geq -\lambda_2 \geq -\lambda_1$$

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Some important ingredients obtained in the last two decades

An impression, that the only persons, who have contributed to this story are Lyapunov, Hodge, Gauss–Manin, Thurston, Masur, Veech, and myself is ... slightly misleading!

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0. Model problem: diffusion in a periodic billiard

1. Teichmüller dynamics (following ideas of B. Thurston)

2. Asymptotic flag of an orientable measured foliation

3. State of the art
- Formula for the Lyapunov exponents
- Strata of quadratic differentials
- Siegel–Veech constant
- Kontsevich conjecture
- Proof: reduction to a combinatorial identity
- Equivalent combinatorial identity
- Invariant measures and orbit closures

∞. What's next?

3. State of the art

Formula for the Lyapunov exponents

Theorem (A. Eskin, M. Kontsevich, A. Z., 2014) *The Lyapunov exponents λ_i of the Hodge bundle $H^1_{\mathbb{R}}$ along the Teichmüller flow restricted to an $\mathrm{SL}(2,\mathbb{R})$-invariant suborbifold $\mathcal{L} \subseteq \mathcal{H}_1(d_1,\dots,d_n)$ satisfy:*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_g \;=\; \frac{1}{12} \cdot \sum_{i=1}^{n} \frac{d_i(d_i+2)}{d_i+1} + \frac{\pi^2}{3} \cdot c_{area}(\mathcal{L})\,.$$

The proof is based on the initial Kontsevich formula $+$ analytic Riemann-Roch theorem $+$ analysis of $\det \Delta_{flat}$ under degeneration of the flat metric.

Theorem (A. Eskin, H. Masur, A. Z., 2003) *For $\mathcal{L} = \mathcal{H}_1(d_1,\dots,d_n)$ one has*

$$c_{area}(\mathcal{H}_1(d_1,\dots,d_n)) = \sum_{\substack{Combinatorial\ types\\ of\ degenerations}} (\text{explicit combinatorial factor}) \cdot$$

$$\cdot \frac{\prod_{j=1}^{k} \operatorname{Vol} \mathcal{H}_1(\text{adjacent simpler strata})}{\operatorname{Vol} \mathcal{H}_1(d_1,\dots,d_n)}\,.$$

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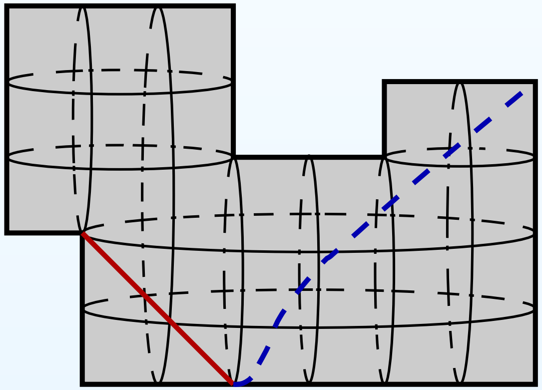
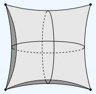
Lyapunov exponents for strata of quadratic differentials

Analogous formula exists for the moduli spaces of slightly more general flat surfaces with holonomy $\mathbb{Z}/2\mathbb{Z}$. They correspond to meromorphic quadratic differentials with at most simple poles. For example, the quadratic differential on the picture below lives in the stratum $\mathcal{Q}(1,1,1,\underbrace{-1,\dots,-1}_{7}) =: \mathcal{Q}(1^3,-1^7)$.





Flat surfaces tiled with unit squares define "integer points" in the corresponding strata. To compute the volume of the corresponding moduli space $\mathcal{Q}_1(d_1,\dots,d_n)$ one needs to compute asymptotics for the number of surfaces with conical singularities $(d_1+2)\pi,\dots,(d_n+2)\pi$ tiled with at most N squares as $N\to\infty$. When $g=0$ this number is the *Hurwitz number* of covers $\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ with a ramification profile, say, as in the picture.

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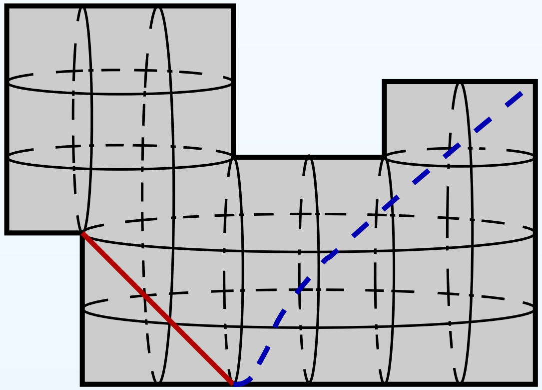

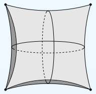

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Lyapunov exponents and alternative expression for the Siegel–Veech constant

Theorem (A. Eskin, M. Kontsevich, A. Z.) *The Lyapunov exponents of the Hodge bundle $H^1_{\mathbb{R}}$ along the Teichmüller flow restricted to a $\mathrm{PSL}(2,\mathbb{R})$-invariant subvariety $\mathcal{L} \subseteq \mathcal{Q}_1(d_1,\dots,d_n)$ satisfy:*

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For $\mathcal{L} = \mathcal{Q}_1(d_1,\dots,d_n)$ one can again express $c_{area}(\mathcal{L})$ in terms of the volumes of the boundary strata, but we do not know yet the values of these volumes except in several cases computed by E. Goujard (2014). However, in genus 0 one can play the following trick.

Corollary. *For any stratum $\mathcal{Q}_1(d_1,\dots,d_n)$ of meromorphic quadratic differentials with at most simple poles **in genus zero** one has*

$$c_{area}(\mathcal{Q}_1(d_1,\dots,d_n)) = -\frac{1}{8\pi^2}\sum_{j=1}^{n}\frac{d_j(d_j+4)}{d_j+2}\,.$$

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Kontsevich conjecture

Let $$v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$, so $v(-1) = 1$ and $v(0) = 2$.

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$$\operatorname{Vol} \mathcal{Q}_1(d_1, \dots, d_k) = 2\pi \cdot \prod_{i=1}^{k} v(d_i)\,.$$

M. Kontsevich conjectured this formula about ten years ago. Using approximate values of Lyapunov exponents which we already knew experimentally, he predicted volumes of the special strata $\mathcal{Q}(d, -1^{d+4})$ and then made an ambitious guess for the general case.

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Proof: reduction to a combinatorial identity

Combining two expressions for $c_{area}(\mathcal{Q}_1(d_1,\dots,d_n))$ we get series of combinatorial identities recursively defining volumes of all strata:

$$\big(\text{explicit combinatorial factor}\big)\cdot\frac{\prod\operatorname{Vol}\big(\text{adjacent simpler strata}\big)}{\operatorname{Vol}\mathcal{Q}_1(d_1,\dots,d_k)}=$$

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It remains to verify that the guessed answer satisfy these identities. The verification is reduced to verifying some combinatorial identities for multinomial coefficients, which is reduced to verifying an equivalent identity for the associated generating functions. The proof uses, however, some nontrivial functional relations for the involved generating functions developing the one discovered by S. Mohanty (1966).

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Equivalent combinatorial identity

$$\frac{6+\sum_{i=1}^{m}\frac{d_i(d_i+1)}{d_i+2}n_i}{\Big(2+(\mathbf{d}+\mathbf{1})\cdot\mathbf{n}\Big)\cdot\Big(3+(\mathbf{d}+\mathbf{1})\cdot\mathbf{n}\Big)\cdot\Big(4+(\mathbf{d}+\mathbf{1})\cdot\mathbf{n}\Big)}\cdot\binom{4+(\mathbf{d}+\mathbf{1})\cdot\mathbf{n}}{\mathbf{n}}$$

$$\stackrel{?}{=}\quad\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}\frac{1}{\Big(1+(\mathbf{d}+\mathbf{1})\cdot\mathbf{k}\Big)\Big(2+(\mathbf{d}+\mathbf{1})\cdot\mathbf{k}\Big)}\cdot\binom{2+(\mathbf{d}+\mathbf{1})\cdot\mathbf{k}}{\mathbf{k}}\cdot$$

$$\cdot\frac{1}{\Big(1+(\mathbf{d}+\mathbf{1})\cdot(\mathbf{n}-\mathbf{k})\Big)\Big(2+(\mathbf{d}+\mathbf{1})(\mathbf{n}-\mathbf{k})\Big)}\cdot\binom{2+(\mathbf{d}+\mathbf{1})\cdot(\mathbf{n}-\mathbf{k})}{\mathbf{n}-\mathbf{k}},$$

where $\mathbf{d}$, $\mathbf{n}$, and $\mathbf{k}$ are nonnegative integer vectors of the same cardinality m, and $\mathbf{1}=\{\underbrace{1,\ldots,1}_{m}\}$; $\mathbf{0}=\{\underbrace{0,\ldots,0}_{m}\}$. Finally, $\binom{l}{\mathbf{k}}:=\binom{l}{k_1,\ldots,k_m,\ l-\mathbf{k}\cdot\mathbf{1}}$.

Invariant measures and orbit closures

Fantastic Theorem (A. Eskin, M. Mirzakhani, 2014). *The closure of any $\mathrm{SL}(2,\mathbb{R})$-orbit is a suborbifold. In period coordinates $H^1(S,\{\text{zeroes}\};\mathbb{C})$ any $\mathrm{SL}(2,\mathbb{R})$-suborbifold is represented by an affine subspace.*

Any ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant measure is supported on a suborbifold. In period coordinates this suborbifold is represented by an affine subspace, and the invariant measure is just a usual affine measure on this affine subspace.

Developement (A. Wright, 2014) *Effective methods of construction of orbit closures.*

Theorem (J. Chaika, A. Eskin, 2014). *For any given flat surface S almost all vertical directions define a Lyapunov-generic point in the orbit closure of $\mathrm{SL}(2,\mathbb{R})\cdot S$.*

Solution of the generalized windtree problem (V. Delecroix–A. Z., 2014). Notice that any "windtree flat surface" S is a cover of a surface S_0 in the hyperelliptic locus $\mathcal{L}$ in genus 1, and that the cycles h and v are induced from S_0. Prove that the orbit closure of S_0 is $\mathcal{L}$. Using the volumes of the strata in genus zero, compute $c_{area}(\mathcal{L})$. Using the formula for $\sum \lambda_i = \lambda_1$ compute λ_1.

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0. Model problem: diffusion in a periodic billiard

1. Teichmüller dynamics (following ideas of B. Thurston)

2. Asymptotic flag of an orientable measured foliation

3. State of the art

∞. What's next?

- What's next?
- Joueurs de billard

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What's next?

- Study and classify all $\mathrm{GL}(2, \mathbb{R})$-invariant suborbifolds in $\mathcal{H}(d_1, \dots, d_n)$. (M. Mirzakhani and A. Wright have recently found an $\mathrm{SL}(2, \mathbb{R})$-invariant subvariety of absolutely mysterious origin.)
- Study extremal properties of the "curvature" of the Lyapunov subbundles compared to holomorphic subbundles of the Hodge bundle. Estimate the individual Lyapunov exponents.
- Prove conjectural formulae for asymptotics of volumes, and of Siegel–Veech constants when $g \to \infty$. (Partial results are already obtained by D. Chen–M. Möller–D. Zagier, 2014–)
- Find values of volumes of $\mathcal{Q}_1(d_1, \dots, d_n)$ in all strata in small genera.
- Express $c_{area}(\mathcal{L})$ in terms of an appropriate intersection theory (in the spirit of ELSV-formula for Hurwitz numbers).
- Study dynamics of the Hodge bundle over other families of compact varieties (some experimental results for families of Calabi–Yau varieties are recently obtained by M. Kontsevich). Are there other dynamical systems, which admit renormalization leading to dynamics on families of complex varieties?

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Group photo from Oberwolfach conference in March 2014

Varvara Stepanova. Joueurs de billard. Thyssen Museum, Madrid