How wild can a group with a quadratic Dehn function be?

Tim Riley

“Boundaries”
Graz

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Euclidean space “enjoys a quadratic isoperimetric function.”
\( \rho \) a loop in a simply connected space \( X \)

\( \text{Area}(\rho) \) is the infimum of the areas of discs spanning \( \rho \).

\[ \text{Area}_X : [0, \infty) \rightarrow [0, \infty] \] is defined by

\[ \text{Area}_X (l) = \sup \{ \text{Area}(\rho) \mid \ell(\rho) \leq l \}. \]
\[ \text{Area}(\Delta) = \# \text{ 2-cells} \]
\[ \mathcal{P} = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle \text{ a finite presentation of a group } \Gamma \]

The presentation 2-complex of \( \mathcal{P} \):

\[ K = \]

\[ \pi_1(K) = \Gamma \]

The universal cover \( \tilde{K} \) is the Cayley 2-complex of \( \mathcal{P} \).

Its 1-skeleton \( \tilde{K}^{(1)} \) is the Cayley graph of \( \mathcal{P} \).
\[ \langle a, b, c, d \mid a^{-1}b^{-1}ab c^{-1}d^{-1}cd \rangle \]

\[ \langle a, b \mid a^{-1}b^{-1}ab \rangle \]

\[ \mathbb{Z}^2 \]

\[ \langle a, b \mid b^{-1}aba^{-2} \rangle \]
$\mathbb{Z}^3 \langle a, b, c \mid [a, b], [b, c], [c, a]\rangle$

A van Kampen diagram
For an edge-loop $\rho$ in the Cayley 2-complex of a finite presentation $\mathcal{P}$, $\text{Area}(\rho)$ is the minimum of $\text{Area}(\Delta)$ over all van Kampen diagrams spanning $\rho$.

The Dehn function $\text{Area}_\mathcal{P} : \mathbb{N} \to \mathbb{N}$ of a finite presentation $\mathcal{P}$ with Cayley 2-complex $\tilde{K}$ is

$$\text{Area}_\mathcal{P}(n) = \max\{\text{Area}(\rho) \mid \text{edge-loops } \rho \text{ in } \tilde{K} \text{ with } \ell(\rho) \leq n\}.$$  

The Filling Theorem. If $\mathcal{P}$ is a finite presentation of the fundamental group of a closed Riemannian manifold $M$ then

$$\text{Area}_\mathcal{P} \preceq \text{Area}_M.$$  

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$f \preceq g$ when $\exists C > 0$ such that $\forall n > 0$, $f(n) \leq Cg(Cn + C) + Cn + C$.

$f \simeq g$ when $f \preceq g$ and $g \preceq f$. 

\[ \langle a, b, c, d \mid a^{-1} b^{-1} a b c^{-1} d^{-1} c d \rangle \]

Area\( (n) \simeq n \)

\[ \langle a, b \mid a^{-1} b^{-1} a b \rangle \]

Area\( (n) \simeq n^2 \)

\[ \langle a, b \mid b^{-1} a b a^{-2} \rangle \]

Area\( (n) \simeq 2^n \)
\[ \text{IP} = \{ \alpha > 0 \mid n \mapsto n^{\alpha} \text{ is } \simeq \text{ a Dehn function} \} \]

The closure of \( \text{IP} \) is

Gromov, Bowditch, N. Brady, Bridson, Olshanskii, Sapir...
"Non–positively curved" groups

Γ a group with finite generating set \( A \)

An asynchronously \( k \)-fellow–travelling linear–length combing is a choice of words \( u_g \) for each \( g \in \Gamma \), such that \( u_g = g \),

\[
\ell(u_g) \leq \text{constant} \cdot d(1, g)
\]

and \( \forall g \in \Gamma, \forall a \in A^{\pm 1} \)

Coning yields a quadratic isoperimetric function:

\[
\text{Area}(n) \leq n^2
\]
Free–by–cyclic groups

\[ F_n \rtimes_\phi \mathbb{Z} \]

e.g. \( F_3 = F(a, b, c) \) with \( \phi : \begin{cases} a &\mapsto a \\ b &\mapsto ab \\ c &\mapsto a^2 c \end{cases} \) is neither CAT(0) nor automatic. [Brady, Bridson, Gersten, Reeves]


To appear as a monograph in the Memoirs of the AMS series. 188 pages!!!
Nilpotent Groups

Example. The class \(c\) free nilpotent group on two generators has Dehn function \(\simeq n^{c+1}\).

Example. The \(5\)-dimensional (class 2) integral Heisenberg group

\[
\begin{pmatrix}
1 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
0 & 1 & 0 & \mathbb{Z} \\
0 & 0 & 1 & \mathbb{Z} \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

has Dehn function \(\simeq n^2\).

Example. The \(m\)-jet bundle, \(J^m(\mathbb{R}^k)\) is a class-\((m + 1)\) nilpotent group and enjoys Euclidean isoperimetric functions for fillings of \(i\)-cycles whenever \(i \leq k\).
Certain semidirect products \( N \rtimes_A \) of a nilpotent and an abelian simply connected Lie group admit quadratic isoperimetric functions.* These include, for \( n \geq 3 \),

\[
\text{Sol}_{2n-1} = \mathbb{R}^n \rtimes \mathbb{R}^{n-1} = \left\{ \begin{pmatrix} e^{t_1} & 0 & \cdots & 0 & x_1 \\ 0 & e^{t_2} & \cdots & 0 & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{t_n} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right| x_i, t_i \in \mathbb{R}, \quad \sum_{i=1}^{n} t_i = 0 \}
\]

which is isometric to a horosphere in \( \text{SL}_4(\mathbb{R})/\text{SO}_4(\mathbb{R}) \).

* In fact, Drutu’s result is more general.
Discrete examples:

**Corollary.** There are polycyclic semi-direct products $\mathbb{Z}^n \rtimes \mathbb{Z}^{n-1}$ (cocompact lattices in $\text{Sol}_{2n-1}$) with exponential growth (so not virtually nilpotent) and quadratic Dehn functions.

[Leuzinger–Pittet]
A finitely presented group whose 3-dimensional integral homology is not finitely generated.*

By John Stallings.

The aim of this note is to provide a counterexample to a conjecture about the niceness of finitely presented groups. This also gives a counterexample to a conjecture about $\pi_2(K)$, where $K$ is a finite complex.

Example. The group $G$ with presentation

$$\{a, b, c, x, y : [x, a], [y, a], [x, b], [y, b], [a^{-1}x, c], [a^{-1}y, c], [b^{-1}a, c]\}$$

with five generators and seven relations has as its 3-dimensional homology group with integer coefficients a not finitely generated group. (Note: $[u, v] = uvu^{-1}v^{-1}$.)

Corollary 1. There is no projective resolution [1] of $\mathbb{Z}$ over $\mathbb{Z}(G)$ which is finitely generated in dimension 3.

Corollary 2. If $K$ is any finite complex with $\pi_1(K) \cong G$, then $\pi_2(K)$ is not finitely generated, even as a module over $\pi_1(K)$.
Bieri–Stallings groups:
\[ \text{Ker} \left( F(a_1, b_1) \times \cdots \times F(a_n, b_n) \rightarrow \mathbb{Z} \right) \]
\[ a_i, b_i \mapsto 1, \quad \forall i \]
are of Type $F_{n-1}$ but not Type $F_n$.

Stallings’ group is the case $n = 3$.

We will work with a presentation of Stallings’ group as an HNN–extension of $F(a, b) \times F(c, d)$:

\[ S = \left\langle a, b, c, d, s \right\rangle \]
The Dehn function of Stallings’ group is...

...at most polynomial [Gersten, 1995]

... $\leq n^5$ [Gersten]

... $\leq n^3$ [Baumslag–Bridson–Miller–Short, 1997]

... $\leq n^{5/2}$ [Elder–R.]

... $\leq n^{7/3}$ [Dison–Elder–R.]

... quadratic [Dison–Elder–R.–Young].
Put words with zero exponent–sum into alternating form:

\[
 u = a^{-1}c^{-1}ab^{-1}ac^{-1}d \\
 \rightarrow a^{-1}aba^{-1}c^{-1}d^{-1}c^{-1}d \\
 \rightarrow ca^{-1}ac^{-1}bc^{-1}ac^{-1} (ac^{-1})^{-2} ac^{-1}ad^{-1}ac^{-1}da^{-1} = \hat{u}
\]

Cost is \( \leq \ell(u)^2 \) and \( \ell(\hat{u}) \leq 4\ell(u) \). Next, divide and conquer very carefully!

The words along the sides of the \( s \)–corridors are alternating.
Thompson’s Group $F$

$$F = \langle a, b \mid (b^a)^b = b^{a^2}, \ (b^{a^2})^b = b^{a^3} \rangle$$

$\cong$ Strictly increasing PL homeomorphisms of $[0, 1]$, that are differentiable except at finitely many dyadic rational numbers and such that all slopes are integer powers of 2.


Theorem. [Brin] Thompson’s group $F$ contains $\cdots ((((\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \mathbb{Z}) \cdots$ and $\cdots (\mathbb{Z} \wr (\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z}))) \cdots$ as subgroups.
Thurston’s Assertion. The Dehn function of $\text{SL}_4(\mathbb{Z})$ is quadratic.

Gromov’s Suggestion. The higher isoperimetric inequalities for $\text{SL}_n(\mathbb{Z})$ concerning filling $k$–spheres by $(k + 1)$–discs agree with those for Euclidean space for $k \leq n - 3$. 

American Institute of Mathematics, 8th–12th September 2008
Does $\text{SL}_4(\mathbb{Z})$ enjoy a polynomial isoperimetric function?

**Yes:** 14

**No:** 1

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Does $\text{SL}_4(\mathbb{Z})$ enjoy a quadratic isoperimetric function?

**Yes:** 10

**No:** 5
[Drutu] Every $\mathbb{Q}$–rank 1 lattice in a semisimple Lie group of $\mathbb{R}$–rank at least 3 is at most asymptotically quadratic. That is, $\forall \varepsilon > 0$, $\exists \ell_\varepsilon > 0$, $\forall \ell \geq \ell_\varepsilon$,

$$\text{Area}(\ell) \leq \ell^{2+\varepsilon}.$$ 

Explicit examples: Hilbert modular groups $\text{PSL}_2(\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers of a totally real field $K$ with $[K : \mathbb{Q}] \geq 3$. 
“I call this a zoo, because I am unable to see any pattern in this bestiary of groups. It would be striking if there existed a reasonable characterization of groups with quadratic Dehn functions, which was more enlightening than saying that they have quadratic Dehn functions.”

Steve M. Gersten

Notes for a CRM Summer School on Groups held at Banff in 1996

Bestiary n., a medieval collection of stories providing physical and allegorical descriptions of real or imaginary animals along with an interpretation of the moral significance each animal was thought to embody.
A graph enjoys the loop–subdivision property when there exists $K$ such that every edge–loop of length $L \gg 0$ can be partitioned into $\leq K$ edge–loops of length $\leq L/2$.

**Theorem.** [Papasoglu] Groups with quadratic Dehn functions enjoy the loop–subdivision property.

**Theorem.** [Gromov] A finitely generated group enjoys the loop–subdivision property if and only if all its asymptotic cones are simply connected.
Example. [Olshanskii–Sapir] The group

\[ \langle a, b, c, k \mid [a, b], [a, c], b^{-1}kb = ka, c^{-1}kc = ka \rangle \]

has Dehn function \( \simeq n^3 \) but none of its asymptotic cones are simply connected. They claim that \( \mathcal{S} \)-machines can be used to produce such an example with Dehn function \( \simeq n^2 \log n \).

Theorem. [R.] The loop–subdivision property implies the filling length function is linear.*

\[
\text{length} = L \leq \text{constant} \cdot L
\]

*This is a slight strengthening of a result of Papasoglu that there is a linear upper bound on the isodiametric function.
Open question. For each \( n \geq 4 \), is there a group which is of Type \( F_{n-1} \) but not of Type \( F_n \) and yet has a quadratic Dehn function?

Possible candidates:

Open question. What are the Dehn functions of the other Bieri–Stallings groups?
The Conjugacy Problem

Given a group $\Gamma$ with some generating set $\mathcal{A}$, find an algorithm which, on input two words on $\mathcal{A}^{\pm 1}$, declares whether or not they represent conjugate elements of $\Gamma$.

The Isomorphism Problem

Given a family of groups, find an algorithm which, on input two groups (e.g. given as finite presentations) of the family, declares whether or not they are isomorphic.
Conjecture. [Rips, Olshanskii–Sapir] Groups with quadratic Dehn function have solvable conjugacy problem.

Theorem. [Olshanskii–Sapir] There is a group with Dehn function $\simeq n^2 \log n$ but no algorithm to decide the conjugacy problem.

Open question: the isomorphism problem for groups with quadratic Dehn function.

cf. the isomorphism problem for CAT(0) groups is open, ... but is solved for torsion–free hyperbolic groups.

[Sela]