Some follow-up from class:

1. If a linear transformation is invertible, it's inverse is <u>automatically</u> linear. So, I didn't need to make a big deal out of that in class! Can you prove this?

   Where this kind of issue does come up is when you are studying <u>continuous</u> functions. There are continuous functions that, set-theoretically, have inverse functions, but the inverse functions need not be continuous. We will discuss some of the subtleties surrounding this issue later this semester.

2. I was completely wrong when I said:

$$\operatorname{rank}(AB) = \operatorname{rank}(BA).$$

Here's a counterexample:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is true that

$$\operatorname{rank}(AB) \le \operatorname{rank}(A)$$
$$\operatorname{rank}(AB) \le \operatorname{rank}(B)$$

and similarly for BA. Can you think about how you would prove this?

In any case, ~~[illegible]~~ to show that an $m \times n$ matrix A, with $m \neq n$, does <u>NOT</u> have both a left & right inverse, this is enough!

Suppose (without loss of generality) that $m < n$. Let B be the $n \times m$ candidate inverse. Then we want $AB = I_{m \times m}$ and $BA = I_{n \times n}$ to be identity matrices. <u>BUT</u>, we know

$$\text{(A)} \quad \left.\begin{array}{l} \operatorname{rank}(A) \le m \\ \operatorname{rank}(A) \le n. \end{array}\right\} \quad \text{think about why!}$$

Thus, since $m < n$, $\boxed{\operatorname{rank}(A) \le m.}$. Similarly, $\boxed{\operatorname{rank}(B) \le m}$. But then, by the inequalities (A),

$$\operatorname{rank}(BA) \le m < n.$$

Also,

$$\operatorname{rank}(I_{n \times n}) = n.$$

Thus, we can't possibly have $BA = I_{n \times n}$!

Think a little bit about the role that the rank-nullity theorem is playing here. If $m \neq n$, then <u>one</u> of AB and BA <u>must</u> have a positive-dimensional kernel.

(3) Note that for a linear transformation

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

the dimension formula tells us that

$$T \text{ is an isomorphism} \iff \ker(T) = \{\vec{0}\}$$
$$\iff \operatorname{im}(T) = \mathbb{R}^n.$$

That is, for a linear transformation from $\mathbb{R}^n$ to itself, surjectivity implies injectivity, and vice versa.

This is NOT true for infinite dimensional vector spaces! Consider

$$T: \mathcal{P}ol(\mathbb{R}) \longrightarrow \mathcal{P}ol(\mathbb{R})$$

defined by

$$T\left(\sum_{i \geq 0} c_i x^i\right) = \sum_{i \geq 0} c_{i+1} x^i.$$

Check that:

a) T is linear;

b) T is surjective; and

c) T is NOT injective.