Additional notes following lecture on 6 Sept. '07 Math 223

It was pointed out that there is a discrepancy between my operation counts from lecture and those discussed in the book, in problem 2.2.11. My operation counts came from a different textbook. Without going into full details, what's clear in either case is that certain algorithms are computationally faster and more effective than others. (In particular you will not need to know, for this course, the formula for the # of operations required!!)

Some brief notes on how to actually compute the LU-factorization:

If we keep track of the row operations needed to put our matrix A into upper triangular form, it is then easy to write down the LU-factorization. So, if

$$\underbrace{F_k \cdot \dots \cdot F_2 \cdot F_1}_{\text{elementary}} \cdot A = U \rightsquigarrow \text{upper-}\Delta \text{ matrix}$$

row ops: assume they're all of the form "Add $x \cdot$(Row j) to Row i". ← necessary to assume no row swaps!

$\rightsquigarrow$ all elementary matrices $E_2(i,j,x)$.

then we have

$$A = F_1^{-1} F_2^{-1} \cdots F_k^{-1} U.$$

Since we needed no row swaps (by assumption), it follows that $F_1^{-1} F_2^{-1} \cdots F_k^{-1}$ is <u>lower triangular</u>, as desired.

Moreover, since $E_2(i,j,x)^{-1} = E_2(i,j,-x)$, it is not hard to compute what that product should be. Can you see what it is?

Example:

Let $A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$ for $a,b,c,d \in \mathbb{R}$.

Let's compute the LU-factorization.

$$\left(\prod_{i=2}^{4} E_2(i,1,-a)\right) A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

Then multiply on the left by

$$\left(\prod_{i=3}^{4} E_2(i,2,-(b-a))\right) \xrightarrow{\text{to get}} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

Finally multiply on the left by $E_2(4,3,-(c-b))$ to obtain

$$U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}.$$

Combining, we get

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a & a-b & 1 & 0 \\ a & a-b & b-c & 1 \end{bmatrix} \cdot \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}}$$

Exercises (optional, but recommended!)

1. Compute L & U for $A = \left[\begin{array}{c|c} 2 & 4 \\ 4 & 11 \end{array}\right]$ & $B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$.

2. Find L & U for $A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}$ for $a,b,c,d,r,s,t \in \mathbb{R}$.

What conditions can you put on a,b,c,d,r,s,t to ensure 4 pivots?