Some notes on quadric surfaces:

DEF A quadric is a subset of $\mathbb{R}^3$ defined by an equation of the form

$$\vec{x} \cdot (A\vec{x}) + \vec{b} \cdot \vec{x} + c = 0$$

where $A \in Mat_{3\times 3}(\mathbb{R})$, $\vec{b} \in \mathbb{R}^3$ and $c \in \mathbb{R}$. $\left(\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$

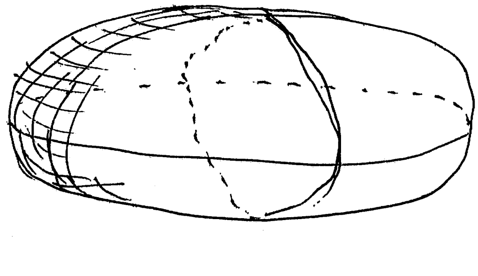
This equation is a polynomial in three variables, where each term has degree at most 2.

A quadric need not be a surface. For example, the equation $x^2 + y^2 + z^2 = 0$ defines a single point, and $x^2 + y^2 = 0$ defines a line. A quadric may also fail to be a manifold. For example, $xy = 0$ defines a union of two planes, the yz-plane and the xz plane, and these intersect along a line. Nevertheless, we can classify quadric surfaces as follows.

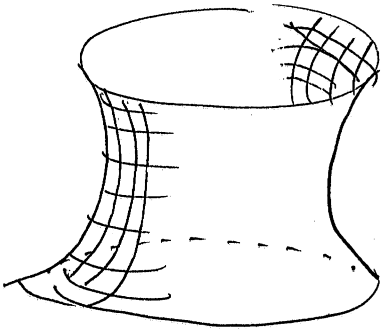
Recall that a rigid motion of $\mathbb{R}^3$ is a rotation about the origin followed by a translation.

Theorem: By applying a rigid motion of $\mathbb{R}^3$, every non-empty quadric where the coefficients are not all zero, can be transformed into one whose cartesian equation is one of the following:

(a) an ellipsoid, $\dfrac{x^2}{p^2} + \dfrac{y^2}{q^2} + \dfrac{z^2}{r^2} = 1$;

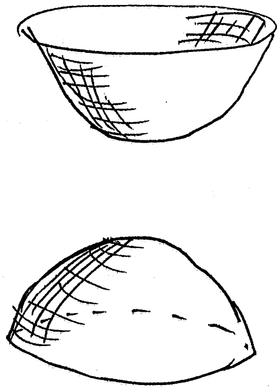


(b) a hyperboloid of one sheet, $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$;



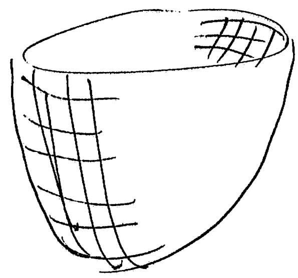
if you cut this with a plane containing the z-axis, you will get a hyperbola.

(c) a hyperboloid of two sheets, $\frac{x^2}{p^2} - \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$;



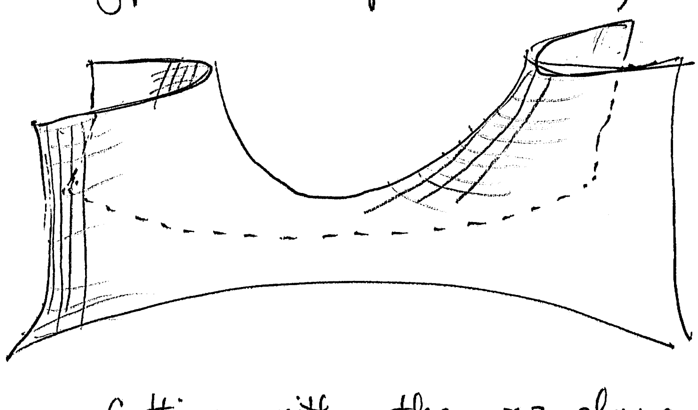
if you cut this with a plane containing the z-axis, you will get a hyperbola.

(d) an elliptic paraboloid, $\frac{x^2}{p^2} + \frac{y^2}{q^2} = z$;



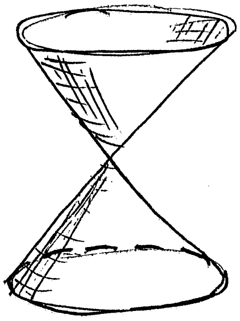
if you slice this with a vertical plane, you will get a parabola.

(e) a hyperbolic paraboloid, $\frac{x^2}{p^2} - \frac{y^2}{q^2} = z$;

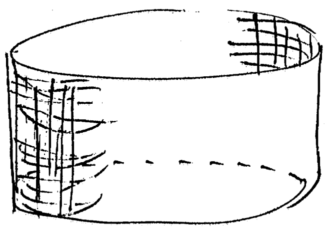


Cutting with the xz-plane or yz-plane gives a parabola.
Cutting with <u>the xy-plane gives a hyperbola.</u>
well, careful ... think about this!

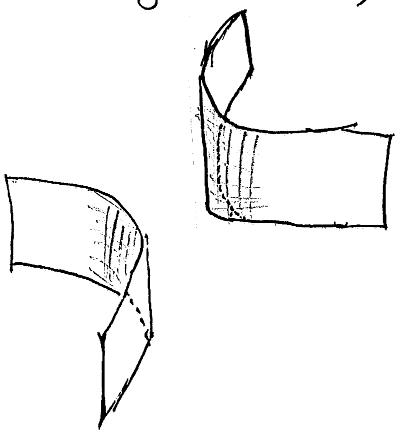
(f) a quadric cone, $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 0$;



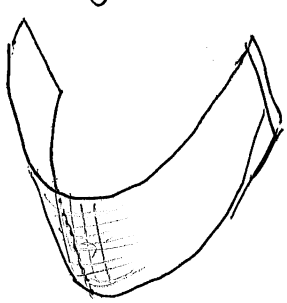
(g) an elliptic cylinder, $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$;



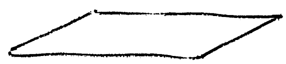
(h) a hyperbolic cylinder, $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$;



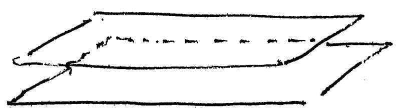
(i) a parabolic cylinder, $\frac{x^2}{p^2} = y$;



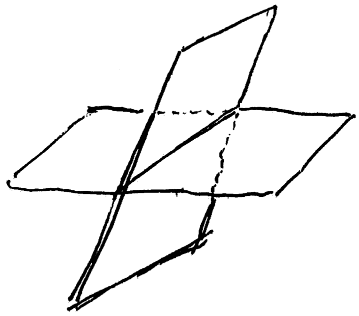
(j) a plane, $x = 0$;



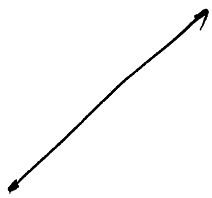
(k) two parallel planes, $x^2 = p^2$;



(l) two intersecting planes, $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 0$;



(m) a straight line, $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 0$; <u>or</u>



(n) a single point, $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 0$.

(In each case, p, q & r are non-zero constants.)

For a proof, see, for example, §4.5 in <u>Elementary Differential Geometry</u> by Andrew Pressley. This book is in the library in Malott Hall, with call number <u>QA641. P68x 2001</u>. It's currently not checked out. If you can't find it there, I have a copy of the book in my office.