Problem Set 3: Solutions.

Book problems (starred) (for rest, ask at recitation.)

<u>2.4.6.</u> Suppose that $\vec{v}$ and $\vec{w}$ are two vectors of an orthonormal basis of $\mathbb{R}^3$. We want to show that the third vector is $\pm\vec{v}\times\vec{w}$.

We know that $\vec{v}$ & $\vec{w}$ span a 2-dimensional subspace of $\mathbb{R}^3$. Thus, there is a 1-dimensional subspace of vectors perpindicular to $\vec{v}$ and $\vec{w}$. Moreover, $\vec{v}\times\vec{w}$ is non-zero and perpindicular to $\vec{v}$ & $\vec{w}$, so any vector in the space perpindicular to $\vec{v}$ & $\vec{w}$ must be a multiple of $\vec{v}\times\vec{w}$. In particular, the third element $\vec{u}$ of the orthonormal basis must be a multiple of $\vec{v}\times\vec{w}$. We know that

$$\begin{aligned}|\vec{v}\times\vec{w}| &= \text{area of parallelogram spanned by } \vec{v} \,\&\, \vec{w}.\\ &= \text{area of a unit square}\\ &= 1.\end{aligned}$$

Thus, we must have $\vec{u} = c\cdot(\vec{v}\times\vec{w})$ for some $c\in\mathbb{R}$ such that

$$\begin{aligned}|\vec{u}| &= |c\cdot\vec{v}\times\vec{w}|\\ &= |c|\cdot|\vec{v}\times\vec{w}|\\ &= |c|.\end{aligned}$$

But we need $|\vec{u}| = 1$, so we must have $c = \pm 1$.
Hence $\boxed{\vec{u} = \pm\vec{v}\times\vec{w}}$, as desired.

2.4.12. We compute, for $A_t = \begin{bmatrix} 2 & t \\ 0 & 2 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_t = \begin{bmatrix} 2 & t \\ 0 & 2 \end{bmatrix}$$

$$A_t^2 = \begin{bmatrix} 4 & 4t \\ 0 & 4 \end{bmatrix}$$

$$A_t^3 = \begin{bmatrix} 8 & 12t \\ 0 & 8 \end{bmatrix}$$

(a) These four vectors are <u>not</u> linearly independent for any t. For example,

$$-4 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4 \cdot \begin{bmatrix} 2 & t \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 4t \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The subspace V_t spanned by these vectors is either one or two dimensional, depending on $t=0$ or $t \neq 0$.

If $t=0$, then $A_t^n = \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}$, and the space V_t consists of all diagonal matrices.

If $t \neq 0$, then I and A_t are a basis for V_t, which is the space of all matrices of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, with $a, b \in \mathbb{R}$.

2.5.8. Let $f: \mathbb{R}^m \to \mathbb{R}^k$ and $g: \mathbb{R}^n \to \mathbb{R}^m$ be linear transformation

<u>True or false</u>: $f \circ g = 0$ implies $\operatorname{img}(g) = \ker(f)$.

<u>FALSE</u>: Consider $f = \vec{0}$ and $g = \vec{0}$ the zero-functions. Then $f \circ g = \vec{0}$ is zero, but $\operatorname{img}(g) = \{\vec{0}\}$ and $\ker(f) = \mathbb{R}^m$.

<u>SALVAGE</u>: What is true is that

$$f \circ g = \vec{0} \quad \text{implies} \quad \operatorname{img}(g) \subseteq \ker(f).$$

1. Prove one of the following statements.
   (a) If $(v_1, \ldots, v_n)$ spans V, then so does the list

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last) the following vector.

PROOF. We need to show that for any $v \in V$, we can write v as a linear combination of $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$. We know that $(v_1, \ldots, v_n)$ spans V, so we may write

$$v = a_1 v_1 + \cdots + a_n v_n. \tag{1}$$

We are looking for coefficients $x_1, x_2, \ldots, x_n$ such that

$$v = x_1(v_1 - v_2) + x_2(v_2 - v_3) + \cdots + x_n v_n.$$

Let's set

$$\begin{aligned} x_1 &= a_1 \\ x_2 &= a_2 + a_1 \\ x_3 &= a_3 + a_2 + a_1 \\ &\vdots \\ x_n &= a_n + a_{n-1} + \cdots + a_1 + a_1. \end{aligned}$$

Then the linear combination $x = x_1(v_1 - v_2) + x_2(v_2 - v_3) + \cdots + x_n v_n$ becomes

$$\begin{aligned} x &= a_1(v_1 - v_2) + (a_2 - a_1)(v_2 - v_3) + \cdots + (a_n - a_{n-1})v_n \\ &= a_1 v_1 - a_1 v_2 + (a_2 + a_1)v_2 - (a_2 + a_1)v_3 + \cdots - (a_{n-1} + \cdots + a_1)v_n + (a_n + \cdots + a_1)v_n \\ &= a_1 v_1 + \cdots + a_n v_n \\ &= v \qquad \text{by equation (1).} \end{aligned}$$

Thus, this linear combination is v, and so this list spans all of V. $\square$

   (b) If $(v_1, \ldots, v_n)$ is linearly independent in V, then so is the list

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last) the following vector.

PROOF. To prove that the list is independent, we must take a linear combination adding up to 0, and prove that all the coefficients are zero. So suppose

$$0 = b_1(v_1 - v_2) + b_2(v_2 - v_3) + \cdots + b_n v_n.$$

Then we may rearrange the right hand side to get

$$\begin{aligned} 0 &= b_1 v_1 - b_1 v_2 + b_2 v_2 - b_2 v_3 + \cdots + b_{n-1} v_{n-1} - b_{n-1} v_n + b_n v_n \\ &= b_1 v_1 + (b_2 - b_1) v_2 + \cdots + (b_n - b_{n-1}) v_n. \end{aligned}$$

But this is a linear combination of the list $(v_1, \ldots, v_n)$, which we have assumed is linearly independent. Thus, the coefficients of <u>this</u> linear combination must all be zero. In other words,

$$\begin{aligned} b_1 &= 0 \\ b_2 - b_1 &= 0 \\ b_3 - b_2 &= 0 \\ &\vdots \\ b_n - b_{n-1} &= 0. \end{aligned}$$

In other words, $b_1 = 0$, $b_2 = b_1$, $b_3 = b_2$, and so on until we conclude that $b_n = b_{n-1} = \cdots = b_1 = 0$. Thus, we have shown that the list $(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$ is linearly independent. □

(c) Suppose that $(v_1, \ldots v_n)$ is linearly independent in V, and $w \in V$. If the list $(v_1 + w, \ldots, v_n + w)$ is linearly dependent, then $w \in \mathrm{span}(v_1, \ldots, v_n)$.

PROOF. By assumption, the list $(v_1 + w, \ldots, v_n + w)$ is linearly dependent, which means that there exist scalars $a_1, \ldots, a_n \in \mathbb{R}$ not all zero so that

$$\begin{aligned} 0 &= a_1(v_1 + w) + a_2(v_2 + w) + \cdots + a_n(v_n + w) \\ &= a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + (a_1 + \cdots + a_n)w. \end{aligned}$$

Because the list $(v_1, \ldots, v_n)$ is linearly independent, the coefficient on w must be non-zero. In other words, $a_1 + \cdots + a_n \neq 0$. Thus, we can rearrange the equation to get

$$-(a_1 + \cdots + a_n)w = a_1 v_1 + \cdots + a_n v_n,$$

and since $-(a_1 + \cdots + a_n) \neq 0$, we can divide by $-(a_1 + \cdots + a_n) \neq 0$ to get

$$w = \frac{a_1}{-(a_1 + \cdots + a_n)} v_1 + \cdots + \frac{a_n}{-(a_1 + \cdots + a_n)} v_n.$$

In other words, w is a linear combination of $(v_1, \ldots, v_n)$, so $w \in \mathrm{span}(v_1, \ldots, v_n)$, which is what we wanted to show. □

2. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ that satisfies $f(a \cdot v) = a \cdot f(v)$ for every $a \in \mathbb{R}$ and $v \in \mathbb{R}^2$, but which is **not** linear.

EXAMPLE. Consider the function

$$f((x, y)) = \sqrt[3]{x^3 + y^3}$$

Then $f(a \cdot (x, y)) = \sqrt[3]{a^3 x^3 + a^3 y^3} = a \cdot \sqrt[3]{x^3 + y^3}$, so this satisfies homogeneity. However, $f((0, 2)) = \sqrt[3]{8}$, $f((2, 0)) = \sqrt[3]{8}$, but $f((0, 2) + (2, 0)) = f((2, 2)) = \sqrt[3]{16} \neq \sqrt[3]{8} + \sqrt[3]{8}$. Thus, this function does not satisfy the additivity property, and hence is not linear. ◇

3. For V and W subspaces of $\mathbb{R}^n$, prove that
(a) $V \cap W$ is a subspace of $\mathbb{R}^n$; and
(b) $V \cup W$ is a subspace of $\mathbb{R}^n$ if and only if $V \subseteq W$ or $W \subseteq V$.

(a) To show that $V \cap W$ is a subspace, we need to show it's closed under scalar multiplication and vector addition.

First, suppose $v \in V \cap W$ and $a \in \mathbb{R}$. Then $v \in V$ and $v \in W$, by definition of intersection, and since those are subspaces, $a \cdot v \in V$ and $a \cdot v \in W$. But ~~this~~ means exactly that $a \cdot v \in V \cap W$.

Second, if $v, w \in V \cap W$, then $v, w \in V$ and $v, w \in W$. Since they are subspaces, $v + w \in V$ and $v + w \in W$, and so $v + w \in V \cap W$.
This proves $\boxed{V \cap W \text{ is a subspace of } \mathbb{R}^n.}$

(b). ($\Leftarrow$) If $V \subseteq W$, then $V \cup W = W$ is a subspace.
If $W \subseteq V$, then $V \cup W = V$ is a subspace.

($\Rightarrow$) We prove this by proving the contrapositive. Suppose $V \nsubseteq W$ and $W \nsubseteq V$. Then we want to show $V \cup W$ is not a subspace of $\mathbb{R}^n$.

Choose $v \in V$ with $v \notin W$, and $w \in W$ with $w \notin V$. (such elements exist by hypothesis)

Then $v + w$ cannot be in $V \cup W$! If it were, $v + w \in V$ or $v + w \in W$. In the first case $v + w, -v \in V$, so $(v+w) + (-v) = w \in V$, contradicting our choice of $w \in W$. In the second, $v + w, -w \in W$, so $(v+w) + (-w) = v \in W$, contradicting our choice of v. Hence, $V \cup W$ is not closed under vector addition, so is not a subspace.

4. We say that two subspaces V and W are **orthogonal** to each other if $v \cdot w = 0$ for every $v \in V$ and $w \in W$. Given an $m \times n$ matrix A, let $R(A)$ denote the subspace of $\mathbb{R}^n$ spanned by the rows of A (thought of, of course, as column vectors!). Prove that $R(A)$ is orthogonal to the kernel of A. Deduce that the image of A is orthogonal to the kernel of A^T.

$R(A) \perp \ker(A)$:

For any vector $v \in \ker(A)$, $A \cdot v = \vec{0}$. But the i^{th} coordinate of $A \cdot \vec{v}$ is the dot product of the i^{th} row of A with $\vec{v}$. Hence, all of these dot products are 0, so $\vec{v}$ is orthogonal to each row of A, and hence to $\text{span}(\text{rows}(A)) = R(A)$.

$\text{Im}(A) \perp \ker(A^T)$:

We deduce this by noting $\text{Im}(A) = \text{Col}(A) = R(A^T)$. Hence, $\text{Im}(A) \perp \ker(A^T)$, by the first part.

5. For subspaces V and W in $\mathbb{R}^n$, define their sum to be

$$V + W = \{v + w \mid v \in V,\ w \in W\}.$$

(a) Prove that $V + W$ is a subspace of $\mathbb{R}^n$.
(b) What is $V + V$?
(c) Prove or give a counterexample to the following statement: If $V_1, V_2, W \subseteq \mathbb{R}^n$ are subspaces such that

$$V_1 + W = V_2 + W,$$

then $V_1 = V_2$.
(d) What can you say about $\dim(V_1 + V_2)$?

5(a). $V+W$ is closed under vector addition, since

$$(v_1 + w_1) + (v_2 + w_2) = \underbrace{\underbrace{(v_1 + v_2)}_{\text{in } V} + \underbrace{(w_1 + w_2)}_{\text{in } W}}_{\text{in } V+W}.$$

and under scalar multiplication, since

$$a \cdot (v + w) = \underbrace{\underbrace{(a \cdot v)}_{\text{in } V} + \underbrace{(a \cdot w)}_{\text{in } W}}_{\text{in } V+W}.$$

Thus, $V+W$ is a vector subspace.

5(b): $V + V = V$. (How do you prove it?!)

5(c): The statement is FALSE. For example, if

$$V_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\} \subseteq \mathbb{R}^2$$

$$V_2 = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \right\} = x\text{-axis}$$

$$W = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \in \mathbb{R}^2 \right\} = y\text{-axis}$$

then $V_1 + W = \mathbb{R}^2 = V_2 + W$. But certainly $V_1 \neq V_2$!

5(d): $\dim(V_1 + V_2) \leq \dim(V_1) + \dim(V_2)$ because any pair of bases, one for V_1 and one for V_2, give a spanning set for $V_1 + V_2$, but it may not be a linearly independent set.

6. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.
   (a) Suppose $\vec{v_1}, \vec{v_2} \in \mathbb{R}^n$ are non-zero, and satisfy $T(\vec{v_1}) = \alpha_1 \vec{v_1}$ and $T(\vec{v_2}) = \alpha_2 \vec{v_2}$ for α_1 and α_2 distinct real numbers. Show that $\{\vec{v_1}, \vec{v_2}\}$ is an independent set.
   (b) Suppose $\vec{v_1}, \dots, \vec{v_k} \in \mathbb{R}^n$ are non-zero, and satisfy $T(\vec{v_i}) = \alpha_i \vec{v_i}$ for $1 \leq i \leq k$, with the α_i distinct real numbers. Show that $\{\vec{v_1}, \dots, \vec{v_k}\}$ is an independent set.
   (c) Exhibit a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ and find a basis $\{\vec{v_1}, \vec{v_2}\}$ of $\mathbb{R}^2$ with $T(\vec{v_1}) = 3\vec{v_1}$ and $T(\vec{v_2}) = 5\vec{v_2}$.
   (d) Exhibit a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that for **no pair** of real numbers (α_1, α_2) is there a basis $\{\vec{v_1}, \vec{v_2}\}$ of $\mathbb{R}^2$ with

$$T(\vec{v_1}) = \alpha_1 \vec{v_1} \text{ and } T(\vec{v_2}) = \alpha_2 \vec{v_2}.$$

6(a): Suppose $(\vec{v}_1, \vec{v}_2)$ is not independent. Then one must be a multiple of the other, $\vec{v}_1 = c \cdot \vec{v}_2$ for some non-zero $c \in \mathbb{R}$. But applying T to this equation, we have

$$\vec{v}_1 = c \cdot \vec{v}_2$$
$$\Rightarrow T(\vec{v}_1) = T(c \cdot \vec{v}_2)$$
$$\Rightarrow \alpha_1 \vec{v}_1 = c \cdot T(\vec{v}_2) = c \cdot \alpha_2 \cdot \vec{v}_2 .$$

Now, subsitituting $v_1 = c\cdot\vec{v}_2$ into the left-hand side, we get

$$\alpha_1 \cdot c \cdot \vec{v}_2 = c\cdot\alpha_2\cdot\vec{v}_2,$$

whence $\alpha_2 = \alpha_1$, contradicting our hypothesis that $\alpha_1 \neq \alpha_2$. Thus, $(\vec{v}_1, \vec{v}_2)$ must be independent.

<u>5(b)</u> We prove this by induction on k. Part <u>5(a)</u> provides the base case. Assume $\vec{v}_1, \ldots, \vec{v}_{k-1}$ are independent, and <u>suppose</u> $\vec{v}_1, \ldots, \vec{v}_k$ is <u>dependent</u>. Then

$$\vec{v}_k = \sum_{i=1}^{k-1} c_i \vec{v}_i \quad (\star)$$

is a linear combination of v_k in terms of $v_1, \ldots, v_{k-1}$, so the coefficients c_i are not all 0, since $\vec{v}_k \neq \vec{0}$.

Applying T, we have

$$T(\vec{v_k}) = T\Big(\sum_{i=1}^{k-1} c_i \vec{v}_i\Big)$$

$$\Rightarrow \alpha_k \vec{v}_k = \sum_{i=1}^{k-1} c_i\, \alpha_i\, \vec{v}_i$$

$$\Rightarrow \alpha_k\Big(\sum_{i=1}^{k-1} c_i \vec{v}_i\Big) = \sum_{i=1}^{k-1} c_i \alpha_i \vec{v}_i \qquad \text{by substituting } (\star).$$

$$\Rightarrow \vec{0} = \sum_{i=1}^{k-1} c_i(\alpha_i - \alpha_k)\vec{v}_i .$$

Thus, we have a linear combination of $\vec{v}_1, \ldots, \vec{v}_{k-1}$, with c_i not all 0 and $\alpha_i - \alpha_k \neq 0$ for all i, since the α_i's are distinct. This contradicts the hypothesis that $\vec{v}_1, \ldots, \vec{v}_{k-1}$ are independent. Thus, our assumption that $\vec{v}_1, \ldots, \vec{v}_k$ are <u>dependent</u> must be false, proving $\vec{v}_1, \ldots, \vec{v}_k$ are in fact <u>independent</u>, as desired.

6(c). Consider $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 5y \end{pmatrix}$.

Then for $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have $T(\vec{v}_1) = 3\vec{v}_1$ and $T(\vec{v}_2) = 5\vec{v}_2$.

6(d). Consider the transformation

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}.$$

To satisfy

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix}$$

we must have

$$\begin{cases} x + y = \alpha x \\ \quad\; y = \alpha y. \end{cases}$$

But $y = \alpha y$ means $\alpha = 1$. Plugging this into the first equation, we have $x + y = x$, ~~so~~ consequently $y = 0$.

But there is no basis of $\mathbb{R}^2$ which has the y-coordinate 0 in both basis vectors.

We know that if there were such (α_1, α_2) and $(\vec{v}_1, \vec{v}_2)$ with $T(\vec{v}_i) = \alpha_i \vec{v}_i$, with $\alpha_1 \neq \alpha_2$ and $\vec{v}_1 \neq \vec{v}_2$ both nonzero, they WOULD BE a basis by 5(a) & $\dim(\mathbb{R}^2) = 2$. Hence, no such pair can exist. $\square$