Induction

Problem 1. Prove *Bernoulli's Inequality* which states:
If $x \geq -1$ then $(1+x)^n \geq 1+nx$ for all natural numbers n.

Problem 2. Prove that

$$1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n = (n-1) \cdot 2^{n+1} + 2.$$

Problem 3. Prove that

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

Problem 4. Prove that

$$\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2} < 1.$$

Problem 5. Prove that the number $11\ldots1$ (consisting of 3^n ones) is divisible by 3^n.

Problem 6. Any two cities in a country are connected by a one-way road. Prove that it is possible to travel through all the cities visiting each city only once.

Problem 7. Let A_1, A_2, …, A_n be $m \times m$ matrices. Prove that the product $A_1 A_2 \ldots A_n$ does not depend on the order of operations. In other words, the result of this expression does not depend on where we put the parentheses.

Problem 8. Prove that $7^{2n} - 48n - 1$ is divisible by 2304 for every natural number n.

Problem 9. Prove that for every natural number n

$$b_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

is the n-th *Fibonacci number* (The sequence of *Fibonacci numbers* is defined in the fallowing way: $b_1 = 1$, $b_2 = 1$, and for $n \geq 3 \quad b_n = b_{n-1} + b_{n-2}$).

Problem 10. How many ways are there to cover the squares of a $2 \times n$ chessboard with dominos?

Problem 11. For the sequence of *Fibonacci numbers* prove that

a) For every $n \geq 1$, b_{3n} is even.
b) For every $n \geq 1$, b_{4n} is divisible by 3.
c) For $n \geq 2$, $b_{n-1}b_{n+1} = b_n^2 + (-1)^n$.