Time limit: 118.75 hours

Your name:

- All answers must be written in complete English sentences. Please show your work and give explanations in complete sentences. Even if your final answer is incorrect, you may get partial credit if you give a coherent explanation your partial progress.
- You may NOT consult the Student Solutions Manual to our textbook. You may consult any other books or your notes, but in the interest of academic integrity, give references to all books that you make use of.
- The ONLY people you may speak to about the exam are Tara Holm and Igors Gorbovickis.
- The exam has **SEVEN** problems. Please carefully write all your final answers, in order they are posed, add this page on the front, and staple all your pages together.

Problem:	Your score:	Total points
1		10 points
2		15 points
3		15 points
4		15 points
5		10 points
6		10 points
7		35 points
Total:		100 points

Academic integrity. As always, you are expected to abide by the Cornell Code of Academic Integrity. This states, "A Cornell student's submission of work for academic credit indicates that the work s the student's own. All outside assistance should be acknowledged, and the student's academic position truthfully reported at all times."

By signing this, I agree to abide by the above guidelines and by the Cornell Code of Academic Integrity.

Your signature:

Good luck!!

1. Let A be an $n \times n$ matrix with $\mathbb{R}$ entries. Determine whether the following are true or false, and justify your response with a proof or counterexample.
   a. If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.
   b. If $\lambda \geq 0$ is an eigenvalue of A^2, then $\sqrt{\lambda}$ is an eigenvalue of A.

2. Identify the 2×2 matrices with $\mathbb{R}$ entries with $\mathbb{R}^4$ by the identification

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Consider the set of 2×2 invertible matrices. This set is usually denoted $GL_2(\mathbb{R})$; the GL stands for "general linear".
   a. Is $GL_2(\mathbb{R})$ an **open** subset in $\mathbb{R}^4$? (Justify your response!)
   b. Is it a **closed** subset in $\mathbb{R}^4$? (Justify your response!)
   c. Show that the function $f : GL_2(\mathbb{R}) \to GL_2(\mathbb{R})$ defined by $f(A) = A^{-1}$ is continuous.

3. Recall that a C^1 function is a function that is differentiable with continuous derivative. Determine whether the following are true or false, and justify your response with a proof or counterexample.
   a. A C^1 function $f : \mathbb{R} \to \mathbb{R}$ that satisfies $f'(x) \neq 0$ for all $x \in \mathbb{R}$ is one-to-one.
   b. A C^1 function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $[Df(x)] \neq 0$ for all $x \in \mathbb{R}^n$ is one-to-one.
   c. A C^1 function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $[Df(x)]$ is invertible for all $x \in \mathbb{R}^n$ is one-to-one.

4. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin(x-y) + y^2 \\ \cos(x+y) - x \end{pmatrix}.$$

   a. Find a global Lipschitz ratio for the derivative of the map F.
   b. Do one step of Newton's method to solve

$$F\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   starting at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
   c. Can you be sure that Newton's method converges?

5. Let $S \subseteq \mathbb{R}^2$ be the unit circle

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1 \right\}.$$

   and let $h : S \to \mathbb{R}$ be a continuous function such that $h\begin{pmatrix} -x \\ -y \end{pmatrix} = -h\begin{pmatrix} x \\ y \end{pmatrix}$ for all

   $\begin{pmatrix} x \\ y \end{pmatrix} \in S$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(t \cdot \overrightarrow{x}) = t \cdot h(\overrightarrow{x}) \text{ for } t \in \mathbb{R} \text{ and } \overrightarrow{x} \in S.$$

   Show that f has directional derivatives at the origin in all directions, but that f is (globally)

   differentiable if and only if there exist $a, b \in \mathbb{R}$ such that $h\begin{pmatrix} x \\ y \end{pmatrix} = ax + by$.

6. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an onto linear transformation, and $g : \mathbb{R}^n \to \mathbb{R}^n$ a C^1 function, with some constant M so that
$$|g(x)| \leq M|x|^2$$
   for every $x \in \mathbb{R}^n$. Define $f = g + T$. Show that f is a C^1 function, and is invertible in a neighborhood of 0.

7. Consider the set $O(n) \subseteq \mathcal{M}at_{n \times n}(\mathbb{R})$ of **orthogonal** matrices: those matrices A whose columns form an orthonormal basis of $\mathbb{R}^n$. Let $S(n) = \{A \in \mathcal{M}at_{n \times n}(\mathbb{R}) \mid A^T = A\}$ be the **symmetric** matrices. Let $A(n)) = \{A \in \mathcal{M}at_{n \times n}(\mathbb{R}) \mid A^T = -A\}$ be the **antisymmetric** matrices.

   a. For a matrix $A \in \mathcal{M}at_{n \times n}(\mathbb{R})$, show that $A \in O(n)$ if and only if $A^T A = I$.
   b. Show that if $A, B \in O(n)$, then $AB \in O(n)$ and $A^{-1} \in O(n)$.
   c. Show that for any matrix $A \in \mathcal{M}at_{n \times n}(\mathbb{R})$, we have $A^T A \in S(n)$.
   d. Define $F : \mathcal{M}at_{n \times n}(\mathbb{R}) \to S(n)$ by $F(A) = A^T A$, so that $O(n) = F^{-1}(I)$. Show that if A is invertible, then $[DF(A)] : \mathcal{M}at_{n \times n}(\mathbb{R}) \to S(n)$ is onto.
   e. Show that $O(n)$ is a manifold embedded in $\mathcal{M}at_{n \times n}(\mathbb{R})$ and that $T_I O(n)$, the tangent space at the identity, is $A(n)$, the space of antisymmetric matrices.
   f. Consider the map $\exp : \mathcal{M}at_{n \times n}(\mathbb{R}) \to \mathcal{M}at_{n \times n}(\mathbb{R})$ defined by

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

   We have seen that this is a convergent sequence. Show that if a matrix $A \in A(n)$ is antisymmetric, then $\exp(A) \in O(n)$ is orthogonal.
   g. For $n = 2$, what is the image of the map $\exp : A(2) \to O(2)$?