Reading. §6.6–6.7.

Problems from the book:

- 3.6.2, 3.6.8
- Page 389 #3.19
- Page 669 #6.10, 6.12, 6.13

Additional problems:

1. A **matroid** is a combinatorial gadget that have applications in many fields of mathematics and beyond. There are many equivalent definitions, though proving equivalence may be difficult. Here are two definitions. Prove that they are equivalent whenever Δ is a **finite** simplicial complex.

   Definition 1. *A* **matroid** *is a simplicial complex Δ that satisfies the* **exchange condition***: for any two simplices $\sigma, \tau \in \Delta$ satisfying $|\sigma| < |\tau|$, there is a vertex $x \in \tau \setminus \sigma$ such that $\sigma \cup \{x\} \in \Delta$.*

   We say that a simplicial complex is **pure** if the maximal simplices all contain the same number of vertices. Given a simplicial complex Δ and a subset $W \subseteq V$ of the vertex set, the **vertex induced subcomplex** on W is the simplicial complex

   $$\Delta(W) = \{\sigma \in \Delta \mid V(\sigma) \subseteq W\},$$

   where $V(\sigma)$ denotes the vertex set of the simplex σ.

   Definition 2. *A pure simplicial complex Δ is a* **matroid** *if the vertex induced subcomplex $\Delta(W)$ is pure for every subset $W \subseteq V$ of the vertex set.*

2. a. Let U be a vector space over $\mathbb{R}$, and let $\Delta(U)$ be the set of sets of linearly independent vectors in U. For example, if $U = \mathbb{R}^n$, then any subset of the standard basis vectors is an element of $\Delta(U)$. Prove that $\Delta(U)$ is a simplicial complex.

   b. Prove that $\Delta(U)$ is a matroid, using one of the above two definitions.