Math 4310

Homework 7

Due **10/24/2012**

Name: ______________________________

Collaborators: ______________________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________________/

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Consider the **elementary matrices** defined in class (and in the text in §12, Example C). These are denoted $P_{i,j}$, $B_{i,j}(\lambda)$, and $D_i(\mu)$ for $\mu \neq 0$.

   (a) What is the inverse of each elementary matrix?

   (b) Given a matrix A, prove that $B_{i,j}(\lambda) \cdot A$ is obtained from A by adding λ times the j^{th} row to the i^{th} row.

2. Which of the following matrices, with entries in the indicated field, are invertible? Please justify your response: find the inverse, or explain why it is not invertible (we do not yet know about determinants for $M_{n\times n}(\mathbb{F})$!).

   (a) $\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$ over $\mathbb{F}_5$

   (b) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ over $\mathbb{F}_3$

   (c) $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ over $\mathbb{R}$

   (d) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ over $\mathbb{F}_2$

3. Show that if $T : V \to W$ is an isomorphism of (finite dimensional) vector spaces over $\mathbb{F}$, then $\dim(V) = \dim(W)$.

4. Suppose that V is an n-dimensional vector space over $\mathbb{F}$, and W an m-dimensional vector space over $\mathbb{F}$. Fix a basis of each vector space, $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$. Consider the map

$$\begin{aligned} \mathcal{L}(V, W) &\to M_{m\times n}(\mathbb{F}) \\ T &\mapsto [T], \end{aligned}$$

where $[T]$ denotes the matrix with i^{th} column the coefficients of $T(v_i)$ expressed as a linear combination of the w_j. Show that this map is an isomorphism of vector spaces.

5. Let V be a 2-dimensional vector space over $\mathbb{R}$. Let $S, T, U \in \mathcal{L}(V, V)$ be given by

$$\begin{array}{lll} S(u_1) = u_1 - u_2 & T(u_1) = u_2 & U(u_1) = 2u_1 \\ S(u_2) = u_1 & T(u_2) = u_1 & U(u_2) = -2u_2, \end{array}$$

where $\{u_1, u_2\}$ is a basis of V. Find the matrices of S, T, and U with respect to the basis $\{u_1, u_2\}$ and with respect to the new basis $\{w_1, w_2\}$ where

$$\begin{aligned} w_1 &= 3u_1 - u_2 \\ w_2 &= u_1 + u_2. \end{aligned}$$

Find invertible matrices X in each case so that $X^{-1}AX = A'$, where A is the matrix of the transformation with respect to the old basis, and A' is the matrix with respect to the new basis.

6. Let $T \in \mathcal{L}(V, V)$. Prove that $T^2 = T \circ T = 0$ if and only if $T(V) \subset n(T)$.

7. For vectors $a = (\alpha_1, \alpha_2)$ and $b = (\beta_1, \beta_2)$ in $\mathbb{R}^2$, define their inner product $\langle a, b\rangle = \alpha_1\beta_1 + \alpha_2\beta_2$. Show that the inner product satisfies the following identities:

   (a) $\langle a, b\rangle = \langle b, a\rangle$.
   (b) $\langle a, b + c\rangle = \langle a, b\rangle + \langle a, c\rangle$.
   (c) $\langle \lambda a, b\rangle = \lambda\langle a, b\rangle$ for all $\lambda \in \mathbb{R}$.
   (d) $||a|| = \sqrt{\langle a, a\rangle}$.
   (e) $a \perp b$ if and only if $\langle a, b\rangle = 0$.
   (f) $a \perp b$ if and only if $||a + b|| = ||a - b||$. Draw a figure to illustrate this statement.

Extended Glossary. In HW4, we defined the **sum** of subspaces $U_1, \ldots, U_n$ of a vector space V over $\mathbb{F}$. We say that V is the **direct sum** of $U_1, \ldots, U_n$ if every vector $v \in V$ can be written *uniquely* as a sum of one vector from each U_i. In your extended glossary this week, please write a formal definition of what it means for V to be the direct sum of some of its subspaces. Then give an example of subspaces of some vector space V so that V is the direct sum, and an example of subspaces where V is not the direct sum. Finally, state and prove a theorem about direct sums. There are other definitions out there of "direct sum". Be careful to use the one I have just described (for example, Wikipedia calls this an "internal direct sum"). For your theorem, you might want to prove that the description I just gave is equivalent to an alternate definition. Or you might want to think about an alternate characterization in the case of just **two** subspaces. If you want more guidance about some interesting statements to try to prove, ask me or the TAs.

As ever, you may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least four examples and two theorems among you. Each one (example and theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.