Math 4310

Homework 2

Due 9/9/15

Name: ______________________________

Collaborators: ______________________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Find the multiplicative inverses of the given elements, or explain why there is not one.
   (a) $[38]$ in $\mathbb{Z}_{83}$
   (b) $[351]$ in $\mathbb{Z}_{6669}$
   (c) $[91]$ in $\mathbb{Z}_{2565}$
2. An element $[a] \in \mathbb{Z}_n$ is **idempotent** if $[a]^2 = [a]$. Find all idempotent elements in $\mathbb{Z}_{10}$ and in $\mathbb{Z}_{13}$.
3. Which of the following functions are well-defined? Justify your answer.
   (a) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f\left(\frac{a}{b}\right) = \frac{3a}{4b}$.
   (b) $g : \mathbb{Q} \to \mathbb{Q}$ defined by $g\left(\frac{a}{b}\right) = \frac{a+2}{b^2}$.
   (c) $h : \mathbb{Z}_8 \to \mathbb{Z}_{12}$ defined by $h([x]_8) = [6x]_{12}$.
   (d) $i : \mathbb{Z}_{12} \to \mathbb{Z}_5$ defined by $i([x]_{12}) = [2x]_5$.
4. Let $M_n(\mathbb{R})$ denote the $n \times n$ matrices with coefficients in $\mathbb{R}$, for $n \geq 1$. This set has two natural operations, matrix addition $+$ and matrix multiplication $\cdot$. For which n is $M_n(\mathbb{R})$ with these operations a field? Please justify your answer.
5. Show that the set of polynomials $\mathbb{Q}[x]$ in one variable with coefficients in $\mathbb{Q}$ is a vector space over the field $\mathbb{Q}$. Did you answer depend on $\mathbb{Q}$, or does your proof generalize to any field $\mathbb{F}$?

Extended Glossary. Please give a definition of a **permutation matrix**. Then give an example of a permutation matrix, an example of a matrix that is not a permutation matrix, and state and prove a theorem about permutation matrices.

You probably saw permutation matrices in Math 2210 or 2940. You can find the definition and some properties described on Wikipedia, but the Wikipedia page does not include the full details of the proofs. Please be sure to write the full details **in your own words** when you write up your extended glossary!

You may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least two examples, two non-examples, and two theorems. Each one (example/non-example/theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.

Some suggestions for facts you might like to prove as your theorem:

(a) Permutation matrices are orthogonal. That is, if A is a permutation matrix, $A \cdot A^T = I$.

(b) The product of two permutation matrices is a permutation matrix.

(c) If P is a permutation matrix and A is any matrix, then $P \cdot A$ has the same entries as A with the (rows/columns?!) permuted. (Check which one!)

(d) If you have already seen the definition of **group**, you could show that the set of $n \times n$ matrices are a group. (You may assume, then, that the reader knows what a group is as well.)