Math 4310

Homework 4

Due 9/23/15

Name: ______________________

Collaborators: ______________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Which of the following sets of vectors in the indicated vector spaces are linearly independent? Please justify your responses.

   A. $(1,1)$, $(2,1)$, and $(1,2)$ in $\mathbb{R}^2$ as a vector space over $\mathbb{R}$.

   B. $f(x) = e^x$ and $g(x) = e^{2x}$ in $\mathscr{F}\mathrm{un}(\mathbb{R},\mathbb{R})$ as a vector space over $\mathbb{R}$.

   C. 1, $\sqrt{2}$, and $\sqrt[4]{2}$ in $\mathbb{R}$ as a vector space over $\mathbb{Q}$.

   D. $1; 1+x; 1+x+x^2; \ldots;$ and $1+x+\cdots+x^n$ in $\mathbb{F}[x]$ as a vector space over $\mathbb{F}$.

2. Find values $a, b \in \mathbb{Q}$ so that $(2, a-b, 1)$ and $(a, b, 3)$ are linearly dependent in $\mathbb{Q}^3$.

3. Compute the dimension of the following vector spaces.

   A. $\mathcal{P}\mathrm{ol}_k(\mathbb{F})$ as a vector space over $\mathbb{F}$.

   B. The vector space of $m \times n$ matrices with entries in a field $\mathbb{F}$, as a vector space over $\mathbb{F}$.

   C. $\left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; \frac{d^n f}{dt^n} = 0 \right\}$ as a vector space over $\mathbb{R}$.

   D. $\left\{ p(x) \in \mathbb{F}_2[x] \;\middle|\; p(x) = p(-x) \right\}$ as a vector space over $\mathbb{F}_2$.

   E. $\ell_\infty(\mathbb{R}) = \left\{ (a_0, a_1, \ldots) \in \ell_0(\mathbb{R}) \;\middle|\; \exists M \in \mathbb{R} \text{ with } |a_i| \le M \ \forall i \right\}$ as a vector space over $\mathbb{R}$.

4. Let $\mathbb{F} = \mathbb{Z}_p$ denote the integers modulo a prime p, and consider the vector space $V = \mathbb{F}^n$ over $\mathbb{F}$. How many bases are there of this vector space? (Your answer should be a formula in terms of p and n.)

Extended Glossary. Given a finite number of subspaces $U_1, \ldots, U_n$ of a vector space V over a field $\mathbb{F}$, we can define their **sum** $U_1 + \cdots + U_n$ to be the set of all sums of vectors $u_1 + \cdots + u_n$ where each $u_i \in U_i$. Please write your extended glossary this week about sums of subspaces. Write out this definition carefully (what I've written is a little informal). Rather than including a non-example, please describe two examples of sums, and then state and prove a theorem about sums of subspaces. You might want to explore the relationship between sums of subspaces and spanning sets of vectors, for example. If you want more guidance about some interesting statements to try to prove, ask me or the TAs.

You may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least two examples, two non-examples, and two theorems. Each one (example/non-example/theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.