Math 4310 | Name: ____________________

Homework 6 | Collaborators: ____________________

Due 10/21/15 | ____________________

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please ***staple*** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ____________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Let V and W be vector spaces over $\mathbb{F}$, and let $\mathsf{LT}(V, W)$ denote the linear transformations from V to W. Prove that $\mathsf{LT}(V, W)$ is a subspace of $\mathscr{F}\mathrm{un}(V, W)$. Suppose that $\dim(V) = 2$, $\dim(W) = 3$, and $v_1, v_2 \in V$ and $w_1, w_2, w_3 \in W$ are bases. Find a basis for $\mathsf{LT}(V, W)$.

2. We will let $\mathsf{LT}(V)$ denote all linear transformations from a vector space V to itself (sometimes called **linear operator**.). Let $T \in \mathsf{LT}(V)$, and recall T^2 denotes the composition $T \circ T$.

   (a) Give an example of a vector space V and a linear operator $T \in \mathsf{LT}(V)$, with $T \neq \mathbb{0}$ and $T \neq \mathbb{1}_V$, so that $T^2 = T$.

   (b) Prove that if $T^2 = T$, then $V = \ker(T) \oplus \ker(T - \mathbb{1}_V)$.

   (c) Prove that if $V = \ker(T) + \ker(T - \mathbb{1}_V)$, then $T^2 = T$.

   (d) Give an example of a vector space V and $T \in \mathsf{LT}(V)$ such that $T^2 = -\mathbb{1}_V$.

3. Let $\mathbb{R}^\infty = \{(\alpha_1, \alpha_2, \ldots) \mid \alpha_i \in \mathbb{R}\}$ be the set of sequences of real numbers. Let $U \subset \mathbb{R}^\infty$ be the subspace of sequences

$$U = \left\{ (\alpha_1, \alpha_2, \ldots) \in \mathbb{R}^\infty \;\middle|\; \alpha_{i+2} = \alpha_i - \alpha_{i+1} \text{ for all } i \right\}.$$

   (You do not need to prove that U is as subspace for this HW, but you should check it for yourself!) Prove that U is finite-dimensional and compute its dimension. Find a **complementary** subspace V so that $U \oplus V = \mathbb{R}^\infty$. (Note: V will not be finite-dimensional!!)

4. Let $C^\infty(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of infinitely-differentiable real-valued functions $f : \mathbb{R} \to \mathbb{R}$.

   (a) Let U denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which vanish at -2 and at π. That is,

   $$U = \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; f(-2) = 0 \text{ and } f(\pi) = 0 \right\}.$$

   (You do not need to prove that U is a subspace.) Prove that the quotient vector space $C^\infty(\mathbb{R})/U$ is finite-dimensional. What is its dimension? (Note: $C^\infty(\mathbb{R})$ is **very** infinite-dimensional!)

   (b) Let W denote the subspace of $C^\infty(\mathbb{R})$ consisting of those functions which "vanish to n^{th} order at 0":

   $$W = \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; f(0) = 0,\ \frac{df}{dx}(0) = 0, \ldots, \text{ and } \frac{d^n f}{dx^n}(0) = 0 \right\}.$$

   Prove that the quotient vector space $C^\infty(\mathbb{R})/W$ is finite-dimensional and find a basis.

5. Compute the following matrix products (where entries are in the indicated field).

   (a) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ -1 & 5 \end{bmatrix}$ where $\mathbb{F} = \mathbb{R}$.

   (b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ where $\mathbb{F} = \mathbb{Z}_2$.

   (c) $\begin{bmatrix} 1 & 2 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ where $\mathbb{F} = \mathbb{Z}_7$.

Extended Glossary. For any vector space V over a field $\mathbb{F}$, define the **dual vector space** V^*. When V is finite dimensional, compute the dimension of V^* by finding a basis for it. Finally state and prove a theorem about dual vector spaces. If you want more guidance about some interesting statements to try to prove, ask me or the TAs.

As ever, you may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least four examples and two theorems among you. Each one (example and theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.