Math 4310

Homework 7

Due 10/28/15

Name: ______________________________

Collaborators: ______________________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear extension of
$$\begin{aligned} T(e_1) &= (-2, 3, 1)\ , \\ T(e_2) &= (3, 0, 1)\ , \text{ and} \\ T(e_3) &= (0, 2, 0) \end{aligned}$$
Write down the matrix of T with respect to the standard basis.

2. Let $\mathbb{F} = \mathbb{Z}_5$, and $T : \mathbb{F}^2 \to \mathbb{F}^2$ the linear transformation whose matrix relative to the standard basis is
$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$
Calculate $T(e_1)$ and $T(e_2)$. Find $v_1, v_2 \in \mathbb{F}^2$ satisfying $T(v_1) = e_1$ and $T(v_2) = e_2$.

3. Let $V = \mathscr{P}o\ell_3(\mathbb{Z}_5)$, and consider the linear transformation $T : V \to V$ defined by
$$T(p(x)) = p(x+1).$$
(You do not need to prove that this is a linear transformation, but you should convince yourself of that fact!) Write down the matrix of T with respect to the usual basis for V.

4. Let $T : \mathbb{F}^3 \to \mathbb{F}^3$ be a linear transformation and $A = (\alpha_{i,j})$ its matrix relative to the standard basis of $\mathbb{F}^3$. Regard the columns of A as vectors in $\mathbb{F}^3$ and call them v_1, v_2, v_3. Show that $\text{Im}(T) = \mathscr{L}(v_1, v_2, v_3)$.

5. Suppose that A is an idempotent matrix. That is, $A^2 = A$. Show that $(\mathbb{I} - A)$ is also idempotent.

6. Let V and W be finite dimensional vector spaces, and $T : V \to W$ a linear transformation. Show that there exist bases of V and of W so that the matrix of T with respect to them is of the form
$$[T] = \begin{bmatrix} \mathbb{I} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix},$$
where $\mathbb{I}$ is an appropriately sized identity matrix, and $\mathbb{0}$ denotes appropriately sized matrices of all 0s.

7. Determine whether the matrices
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
represent the same linear transformation with respect to different ordered pairs of bases of $\mathbb{F}^3$, for $\mathbb{F} = \mathbb{Z}_7$.

8. A vector $v \in V$ is called a **fixed point** of a linear transformation $T : V \to V$ if $T(v) = v$. It is called an **anti-fixed point** if $T(v) = -v$. Consider the transformations represented by the following matrices (with respect to the standard basis), and determine the fixed points and anti-fixed points of each.
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

9. Put the matrix $\begin{bmatrix} 2 & 0 & 3 & 2 & 6 \\ 5 & 4 & 0 & 0 & 0 \\ 3 & 0 & 4 & 6 & 5 \\ 3 & 1 & 0 & 5 & 1 \end{bmatrix} \in M_{4\times 5}(\mathbb{Z}_7)$ into reduced row echelon form.

10. For which values of α does the following system of equations have a solution (over $\mathbb{R}$)?
$$\begin{array}{rcrcrcl} & & y & + & z & = & 2, \\ x & + & \alpha y & + & z & = & 2 \\ x & + & y & & & = & 2 \end{array}$$

Extended Glossary. In your extended glossary this week, please give a definition of the **transpose** of a matrix, and of a **symmetric matrix** and a **skew-symmetric matrix**. Then give an example of each a symmetric and a skew-symmetric matrix, and an example that is neither symmetric nor skew-symmetric. Finally, state and prove a theorem about symmetric and/or skew-symmetric matrices. If you want more guidance about some interesting statements to try to prove, ask me or the TAs.

As ever, you may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least four examples and two theorems among you. Each one (example and theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.