Math 4310

Homework 8

Due 11/4/15

Name: ______________________________

Collaborators: ______________________________

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Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises ______________________ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Find the characteristic polynomial, eigenvalues and eigenvectors, and diagonalize *if possible* each of the following linear transformations:
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (y, -x)$.
   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (x + 2y, 2y + x, z)$.
   (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (3x - z, -x + y + 2z, 4z)$.
   (d) $T : (\mathbb{Z}_7)^2 \to (\mathbb{Z}_7)^2$, $T(x, y) = (x + 2y, 2x + y)$.
2. Prove that if a linear transformation $T : V \to V$ is nilpotent, then all its eigenvalues are 0.
3. Let $T : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that T^2 has a positive eigenvalue. Show that T has a real eigenvalue.
4. Suppose that $T : V \to V$ is an isomorphism. What is the relationship between the eigenvalues of T and of T^{-1}?
5. Let $T : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ be the linear transformation given by $T(A) = A^{tr}$ (the **transpose**).
   (a) Find the characteristic polynomial of T.
   (b) Determine the eigenvalues and eigenspaces of T
   (c) Is T diagonalizable? If so, diagonalize it. If not, prove it is not.

For $T : V \to V$ a linear transformation, you will need the following definition for #6. We say that a subspace $W \subset V$ is T-**invariant** if $T(w) \in W$ for each $w \in W$.

6. Let $C^\infty(\mathbb{R}, \mathbb{C})$ be the vector space of complex-valued functions on $\mathbb{R}$ that are infinitely differentiable. Let V be the subset of functions $f \in C^\infty(\mathbb{R}, \mathbb{C})$ satisfying the differential equation $\frac{d^2f}{dx^2} = -f$:

$$V = \left\{ f \in C^\infty(\mathbb{R}, \mathbb{C}) \;\middle|\; \frac{d^2f}{dx^2} = -f \right\}.$$

   (a) Prove that V is a subspace of $C^\infty(\mathbb{R}, \mathbb{C})$.

   (b) If you take a course on differential equations, you'll learn how to prove that the space of solutions V is at most 2-dimensional, from the form of the differential equation $f' = -f$. However, since this is a linear algebra course, just trust me on this, and assume without proof that $\dim(V) \leq 2$.

   Prove that the functions $\sin(x)$ and $\cos(x)$ both lie in V, and moreover that $\sin(x)$ and $\cos(x)$ form a basis for V.

   (c) For the linear transformation $D : C^\infty(\mathbb{R}, \mathbb{C}) \to C^\infty(\mathbb{R}, \mathbb{C})$ defined by $D(f) = \frac{df}{dx}$, show that V is a D-invariant subspace.

   (d) Now consider D as a linear transformation $D : V \to V$. Find a basis of V consisting of eigenvectors for D. What are their eigenvalues?

Extended Glossary. In your extended glossary this week, please give a definition of the **rank** of a linear transformation, give two examples where you compute the rank (one can be very straight forward, but the other should include at least a little computation). Finally, state and prove a theorem about the rank of a linear transformation. If you want more guidance about some interesting statements to try to prove, ask me or the TAs.

As ever, you may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least four examples and two theorems among you. Each one (example and theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.