Math 4310

Homework 9

Due 11/11/15

Name: _____

Collaborators: _____

_____

Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please **staple** *this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.*

GRADES

Exercises _____ / 50

Extended Glossary

Component	Correct?	Well-written?
Definition	/6	/6
Example	/4	/4
Non-example	/4	/4
Theorem	/5	/5
Proof	/6	/6
Total	/25	/25

Exercises.

1. Compute the determinants of the following matrices.

   (a) All 3×3 permutation matrices in $M_{3\times 3}(\mathbb{F})$.

   (b) $\begin{bmatrix} 1 & 0 & 3 & 4 & 1 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 4 & 2 \end{bmatrix} \in M_{5\times 5}(\mathbb{F}_5)$. (HINT: You probably want to row reduce!)

2. Let $\alpha \in \mathbb{F}$ be non-zero, and consider the matrix $A = \begin{bmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & \alpha \end{bmatrix} \in M_{4\times 4}(\mathbb{F})$. Show that A is invertible, and find A^{-1}. What is A^{-1} when $\alpha = 3 \in \mathbb{F}_7$?

3. Let $\mathbb{F}$ be a field, $n \geq 1$ a positive integer, and $A, B \in M_{n\times n}(\mathbb{F})$. Show that A and B are both invertible if and only if
$$\begin{bmatrix} A & \mathbb{O} \\ \mathbb{O} & B \end{bmatrix} \in M_{2n\times 2n}(\mathbb{F})$$
is invertible. (As usual, $\mathbb{O}$ denotes the $n \times n$ matrix with all entries 0.)

4. Let U and V be vector spaces over $\mathbb{F}$. Their **direct product** is the set of ordered pairs

$$U \times V = \{(u, v) \mid u \in U \text{ and } v \in V\}.$$

It is not hard to show that this is again a vector space over $\mathbb{F}$, where addition and scalar multiplication are coordinate-wise. (You do not need to prove that here.)

(a) Compute the dimension of $U \times V$ (in terms of the dimensions of U and of V).

(b) Let $S : U \to U$ and $T : V \to V$ be linear transformations. Define $S \times T \in \mathsf{LT}(U \times V, U \times V)$ by $S \times T((u, v)) = (S(u), T(v))$. Prove that if $\lambda \in \mathbb{F}$ is an eigenvalue for $S \times T$, then it's an eigenvalue for S or for T (or both).

(c) Now suppose that U and V are finite dimensional. Prove that $\det(S \times T) = \det(S) \cdot \det(T)$.

5. Let $v_1, \dots, v_n$ be a basis for V. We say that an operator $T \in \mathsf{LT}(V, V)$ is **upper-triangular with respect to the basis** $v_1, \dots, v_n$ if $T(v_i) \in \mathscr{L}(v_1, \dots, v_i)$ for all $i = 1, \dots, n$. Assume that T is upper-triangular with respect to the basis $v_1, \dots, v_n$, so for each i we can write $T(v_i) = d_i v_i + w_i$ for some $d_i \in \mathbb{F}$ and $w_i \in \mathscr{L}(v_1, \dots, v_{i-1})$.

(a) Prove that $\det(T) = d_1 \cdot d_2 \cdots d_n$.

(b) Prove that each number d_i is an eigenvalue of T. Note that the vectors v_i are almost certainly not eigenvectors of T!

(HINT: I do not think a direct approach is best here. First think about how you would prove it when $d_i = 0$, then reduce the general case to this.)

6. Let $\mathbb{R}^\infty = \{(\alpha_1, \alpha_2, \dots) \mid \alpha_i \in \mathbb{R}\}$ be the set of sequences of real numbers. Consider the shift operator $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $T((\alpha_1, \alpha_2, \dots)) = (0, \alpha_1, \alpha_2, \dots)$.

(a) The operator $T + \mathbb{I}$ is given by

$$(T + \mathbb{I})((\alpha_1, \alpha_2, \alpha_3, \dots)) = (\alpha_1, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \dots).$$

Find an inverse $(T + \mathbb{I})^{-1}$ for this operator.

(b) For which values of $\lambda \in \mathbb{R}$ is the operator $(T - \lambda\mathbb{I})$ non-invertible? Justify your response!

(c) What are the eigenvalues of T?

(d) Explain the discrepancy between your answers to (b) and (c).

7. Let V and W be finite dimensional vector spaces, $V^* = \mathsf{LT}(V, \mathbb{F})$ and $W^* = \mathsf{LT}(W, \mathbb{F})$ their dual spaces, and $S : V \to W$ a linear transformation. Define $S^{tr} : W^* \to V^*$ (pronounced **S-transpose**) as follows. If $f \in W^*$ is a linear transformation $W \to \mathbb{F}$, then $S^{tr}(f) : V \to \mathbb{F}$ is defined by $S^{tr}(f)(v) = f(S(v))$.

   (Note: $S^{tr}(f)$ is linear! You don't have to prove it, but you should think about why it's true.)

   (a) Prove that S^{tr} is a linear transformation $W^* \to V^*$.
   (The reason S^{tr} is pronounced "S-transpose" is that if you choose bases of V and W, and then take the corresponding dual bases of W^* and V^*, the matrices of S and S^{tr} with respect to those bases are transposes of one another. You do not need to prove this, but do think about why it's true!)

   (b) Let $\text{Transpose} : \mathsf{LT}(V, W) \to \mathsf{LT}(W^*, V^*)$ be defined by $\text{Transpose}(S) = S^{tr}$. Prove that Transpose is a linear transformation.

   (c) Prove that $\mathbb{0}^{tr} = \mathbb{0}$, and when $V = W$ that $\mathbb{1}_V^{tr} = \mathbb{1}_{V^*}$

   (d) If $S \in \mathsf{LT}(V, W)$ and $R \in \mathsf{LT}(W, U)$, prove that $(R \circ S)^{tr} = S^{tr} \circ R^{tr}$.

Extended Glossary. Please give a definition of an **orthogonal matrix**. Then give an example of an orthogonal matrix, an example of a matrix that is not an orthogonal matrix, and state and prove a theorem about orthogonal matrices.

You probably saw orthogonal matrices in Math 2210 or 2940. You can find the definition and some properties described on Wikipedia, but the Wikipedia page does not include the full details of the proofs, and includes some facts that are probably impossible for you to prove at this point. If you want some suggestions of statements you might try to prove, ask for some hints from me or from the TAs.

As ever, you may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least four examples and two theorems among you. Each one (example and theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.