

Extended Glossary. Please give a definition of a **rational number**. Then give an example of a rational number, an example of a number that is not rational, and state and prove a theorem about rational numbers.

You may work in groups, but please write up your solutions **yourself**. If you do work together, your group should come up with at least two examples, two non-examples, and two theorems. Each one (example/non-example/theorem) should be included in some group member's extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.

Your grade:

| Component | Correct? | Well-written? |
|-------------|----------|---------------|
| Definition | /6 | /6 |
| Example | /4 | /4 |
| Non-example | /4 | /4 |
| Theorem | /5 | /5 |
| Proof | /6 | /6 |
| Total | /25 | /25 |

Fields arise in the number systems that we learn in elementary mathematics. Addition and multiplication of real numbers are familiar operations, and satisfy the commutative, associative and distributive laws. The natural numbers that we learn as children when we learn to count do not form a field because we must have additive inverses. The additive inverse of a number is its negative, and a negative number is not a natural number.

The set \mathbb{Z} of integers does include additive inverses, but does not have multiplicative inverses. The multiplicative unit is the number 1. The number $2 \in \mathbb{Z}$ does not have a multiplicative inverse in \mathbb{Z} : there is no other integer m such that $2 \cdot m = 1$. Indeed, this equation holds only when $m = \frac{1}{2} \notin \mathbb{Z}$. The number $\frac{1}{2}$ is a rational number, the first set of numbers that we encounter that do form a field.

Definition 1. A **rational number** is any number that may be expressed as a quotient $\frac{a}{b}$ of two integers $a, b \in \mathbb{Z}$ where the denominator is non-zero $b \neq 0$. The rational numbers are a set

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

They are endowed with the addition and multiplication of real numbers, and so addition and multiplication are commutative, associative and distributive.

Example 2. We have already seen an example of a rational number, the fraction $\frac{1}{2}$. Any integer $a \in \mathbb{Z}$ is also a rational number since we implicitly have

$$a = \frac{a}{1}.$$

We will (almost always) write our fractions in reduced form, and when the denominator is 1 as above, we will simply write the fraction as the numerator a . \diamond

Example 3. A real number that is **not** rational is called **irrational**. We will now show that the number $\sqrt{2}$ is irrational. This number $\sqrt{2}$ is defined to be the number that

$$(\sqrt{2})^2 = 2.$$

We will prove that it is irrational by contradiction: we assume that it is rational, and then derive a logical contradiction.

Suppose that $\sqrt{2}$ is rational, and write it as a fraction in reduced form, that is

$$\sqrt{2} = \frac{a}{b},$$

where $a, b \in \mathbb{Z}$, $b \neq 0$, and there is no other integer $c \geq 1$ that divides both a and b . We now manipulate our equation

$$(4) \quad \sqrt{2} = \frac{a}{b} \quad ;$$

$$(5) \quad (\sqrt{2})^2 = \left(\frac{a}{b}\right)^2 \quad \text{by squaring both sides ;}$$

$$(6) \quad 2 = \frac{a^2}{b^2} \quad \text{since } (\sqrt{2})^2 = 2 \text{ and squaring the right-hand side; and}$$

$$(7) \quad 2b^2 = a^2 \quad \text{by multiplying through by } b^2.$$

Now we notice that a^2 must be an even number. But if the square of an integer is even, that integer itself must be even as well. This means that $a = 2d$ for some other integer $d \in \mathbb{Z}$. Thus, $a^2 = (2d)^2 = 4d^2$. Plugging this back into equation (??), we get

$$(8) \quad 2b^2 = 4d^2 \text{ and so}$$

$$(9) \quad b^2 = 2d^2 \text{ by dividing through by 2.}$$

Now we see that b^2 is an even number, and so as above, b must also be even.

Thus we have shown that if $\sqrt{2} = \frac{a}{b}$, then a and b must both be even integers. This contradicts our assumption that $\frac{a}{b}$ is a reduced fraction. Hence our original hypothesis that $\sqrt{2}$ is rational must be incorrect, and so we conclude that $\sqrt{2}$ is irrational. \diamond

We now prove that the rational numbers are indeed a field.

Theorem 10. *The set of rational numbers \mathbb{Q} together with the numerical operations $+$ and \cdot is a field.*

Proof. To check that a set with two operations $+$ and \cdot is a field, we must verify the seven conditions listed in our text [?, p. 8]. We use the same numbers as in the reference.

1. We know from the properties of $+$ and \cdot as operations on real numbers that $+$ and \cdot are *commutative* operations.
2. We know from the properties of $+$ and \cdot as operations on real numbers that $+$ and \cdot are *associative* operations.
3. We know from the properties of $+$ and \cdot as operations on real numbers that $+$ and \cdot satisfy the *distributive law*.
4. The *additive unit* in \mathbb{Q} is zero $0 \in \mathbb{Q}$. We know from the elementary properties of real numbers that

$$0 + \frac{a}{b} = \frac{a}{b}$$

for any fraction $\frac{a}{b} \in \mathbb{Q}$.

5. Let $\frac{a}{b} \in \mathbb{Q}$. Then $\frac{-a}{b}$ is also a rational number, and

$$\frac{a}{b} + \frac{-a}{b} = \frac{a + (-a)}{b} = \frac{0}{b} = 0.$$

Thus every rational number has an *additive inverse*.

6. The *multiplicative unit* in \mathbb{Q} is one $1 \in \mathbb{Q}$. Indeed,

$$1 \cdot \frac{a}{b} = \frac{1 \cdot a}{b} = \frac{a}{b}$$

for any $\frac{a}{b} \in \mathbb{Q}$.

7. Suppose that $\frac{a}{b} \in \mathbb{Q}$ is non-zero. This means that $a \neq 0$ and $b \neq 0$. But then the fraction $\frac{b}{a}$ is also a rational number, and

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{a \cdot b}{b \cdot a} = \frac{a \cdot b}{a \cdot b} = 1.$$

Thus every non-zero rational number has a *multiplicative inverse*.

We have checked the seven defining conditions, and so we conclude that \mathbb{Q} is a field. \square

REFERENCES

- [1] C. Curtis, *Linear Algebra. An Introductory Approach*. Corrected reprint of the 1984 fourth edition. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1993.
- [2] *Rational number*. Web http://en.wikipedia.org/wiki/Rational_number. 15 August 2012.