1. (a)

$$\begin{aligned}(123)(345) &= (12345)\\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}\\ &= (15)(14)(13)(12)\end{aligned}$$

Sign$= +1$

(b)

$$\begin{aligned}(125)(234)(56) &= (123456)\\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}\\ &= (16)(15)(14)(13)(12)\end{aligned}$$

Sign$= -1$

(c)

$$\begin{aligned}(15)(152)(278) &= (2785)\\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 7 & 3 & 4 & 2 & 6 & 8 & 5 \end{pmatrix}\\ &= (25)(28)(27)\end{aligned}$$

Sign$= -1$

2. (Rotman #2.33) We just need to pick α, β, γ such that α and β are disjoint, α and γ are disjoint, but β and γ are not disjoint. For example $\alpha = (12), \beta = (34), \gamma = (35)$

3. Let $\alpha = (ij)$ and $\beta = (a_1 a_2 \cdots a_r)$. Either

   (a) $\{i, j\}$ is disjoint from $\{a_1, \ldots, a_r\}$, in which case α commutes with β.

   (b) $i \in \{a_1, \ldots, a_r\}$ but $j \notin \{a_1, \ldots, a_r\}$

   Say $i = a_i$. Then

   $$(a_i j)(a_1 \cdots a_r)(a_i j) = (a_1 \cdots a_{i-1} j a_{i+1} \cdots a_r)$$

   So this has the effect of replacing a_i with j.

   (c) Both $i, j \in \{a_1, \ldots, a_r\}$

   Say $i = a_i, j = a_j$. WLOG assume $i < j$

   $$(a_i a_j)(a_1 \cdots a_r)(a_i a_j) = (a_1 \cdots a_{i-1} a_j a_{i+1} \cdots a_{j-1} a_i a_{j+1} \cdots a_r)$$

   This has the effect of swapping a_i and a_j.

4. (i) An r-cycle is of the form $(a_1 a_2 \cdots a_r)$. There are n choices for a_1, $n-1$ choices for a_2, …, $n-r+1$ choices for a_r. Observe that r such notions describe the same permutation in S_n,

   e.g. $(1234) = (2341) = (3412) = (4123)$. These 4 cycles describe the same permutation
   e.g. $(a_1 a_2 \cdots a_r) = (a_2 \cdots a_r a_1) = \cdots = (a_r a_1 \cdots a_{r-1})$

   We thus get the desired formula

   $$\text{number of } r\text{-cycles} = \frac{1}{r}\left(n(n-1)\cdots(n-r+1)\right)$$

(ii) Such a permutation is of the form

$$\alpha = (a_{11}a_{12}\cdots a_{1r})(a_{21}a_{22}\cdots a_{2r})\cdots(a_{k1}a_{k2}\cdots a_{kr})$$

These are disjoint r-cycles. There are

n choices for a_{11}
$n-1$ choices for a_{12}
$\cdots$
$n-kr+1$ choices for a_{kr}

Again, for each r-cycle, we have over-counted by a factor of r. There are k r-cycles, so we have to divide by r k-times. Finally, observe that for disjoint r-cycles, the order we write them in is irrelevant,

e.g. $(123)(456)(789) = (123)(789)(456) = (789)(123)(456) = \cdots$

There are $k!$ different possible permutations, thus giving the desired formula for the number of permutations in S_n which are represented by a product of k disjoint r-cycles as

$$\frac{1}{k!}\frac{1}{r^k}[n(n-1)\cdots(n-kr+1)]$$

5. (a) If $\alpha \in A_n$ then α factors into an even number of transpositions. Therefore for $\alpha, \beta \in A_n$ write

$$\alpha = \tau_1 \cdots \tau_{2r}$$
$$\beta = \sigma_1 \cdots \sigma_{2s}$$

where τ_i, σ_i are transpositions. Clearly $\alpha\beta = \tau_1 \cdots \tau_{2r} \cdot \sigma_1 \cdots \sigma_{2s}$ is a product of $2(r+s)$ transpositions, and thus is in A_n. We compute the inverse

$$\alpha^{-1} = \tau_{2r} \cdots \tau_2 \tau_1$$

which again is in A_n.

(b) Here is a general r-cycle,

$$\rho = (a_1 a_2 \cdots a_r)$$

It decomposes into transpositions as,

$$\rho = (a_1 a_r)(a_1 a_{r-1}) \cdots (a_1 a_2)$$

a product of $r-1$ transpositions. Thus $\rho \in A_n$ iff r is odd.

(c) Let $|\sigma|$ denote the number of transpositions that a permutation σ decomposes into. To see this let's observe first that $|\sigma| = |\sigma^{-1}|$, an inverse decomposes into the same number of transpositions (see part (a) where we compute the inverse as a product of transpositions). Therefore the *commutator* $\sigma\tau\sigma^{-1}\tau^{-1}$ decomposes into $|\sigma| + |\tau| + |\sigma^{-1}| + |\tau^{-1}| = 2(|\sigma| + |\tau|)$ transpositions and thus is in A_n.

(d) Consider the function

$$f : S_n \to S_n$$
$$\sigma \mapsto (12) \cdot \sigma$$

This is a bijection (indeed, f is its own inverse!). Now observe that if f maps even permutations to odd ones, and odd permutations to even ones. Therefore, since f is a bijection, the number of even permutations is equal to the number of odd permutations and we therefore get that

$$|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$$

(e) id: there is one identity permutation and it's cycle shape is (1,1,1,1,1)
(3,1,1): e.g. (123), there are 20
(2,2,1): e.g. (12)(34), there are 15
(5): e.g. (12345), there are 24.