Math 4320

Homework 2

Due 2/4/15

Name: ____________________

Collaborators: ____________________

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Please print out this page. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please **staple** *this cover sheet and your solutions together and hand in your homework in class.*

GRADE

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(For instructors's use only)

Exercises.

1. For the following permutations, please give the disjoint cycle decomposition, the two line notation, a transposition factorization and the sign of the permutation. (Note that none of these depend on which S_n the permutation is in!)

   (a) $(1,2,3)(3,4,5)$

   (b) $(1,2,5)(2,3,4)(5,6)$

   (c) $(1,5)(1,5,2)(2,7,8)$

2. (From the book, p. 124, # 2.33) Give an example of $\alpha, \beta, \gamma \in S_5$, none of which is the identity (1), with $\alpha\beta = \beta\alpha$ and $\alpha\gamma = \gamma\alpha$, but with $\beta\gamma \neq \gamma\beta$.

3. Given a transposition $\alpha = (i, j)$ and a cycle $\beta \in S_n$, what effect does $\alpha^{-1}\beta\alpha$ have on β?

   (Hint: the answer will depend on whether only i, only j, or i and j are letters in the cycle β!)

4. (From the book, p. 124, # 2.24) This problem has you count the number of cycles and permutations of particular cycle type. If you like this kind of problem, I have a kind of crazy but fun puzzle for you, come ask me.

   (i) If $1 < r \leq n$, prove that there are

   $$\frac{1}{r}\Big[n(n-1)\cdots(n-r+1)\Big]$$

   r-cycles in S_n.

   (ii) If $kr \leq n$, where $1 < r \leq n$, prove that the number of permutations $\alpha \in S_n$ where α is a product of k disjoint r-cycles is

   $$\frac{1}{k!}\frac{1}{r^k}\Big[n(n-1)\cdots(n-kr+1)\Big].$$

5. Consider the subset $A_n \subset S_n$ consisting of just those permutations whose factorization into transpositions contains an **even** number of transpositions. These are known as the **even permutations**.

   (a) Show that for $\alpha, \beta \in A_n$, we also have $\alpha\beta \in A_n$ and $\alpha^{-1} \in A_n$. We say that A_n is **closed under composition** and **closed under inversion**.

   (b) Show that an r-cycle is an even permutation if and only if r is odd.

   (c) For any elements $\sigma, \tau \in S_n$, show that $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$.

   (d) For $n \geq 2$, prove that the number of even permutations in S_n is $\frac{1}{2}n!$.

   (e) Without writing down all 60 elements of A_5, describe all the possible shapes of the permutations (the number and lengths of their disjoint cycles) and how many of each type there are.