Math 4320

Homework 3

Due 2/11/15

Name: ______________________

Collaborators: ______________________

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Please print out this page. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please **staple** *this cover sheet and your solutions together and hand in your homework in class.*

GRADE
______________________ /
(For instructors's use only)

Exercises.

1. **True / False**. Determine whether the following statements are true or false. Justify your response with a proof or counterexample.

   (a) The function $e : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $e(m, n) = m^n$ is associative.

   (b) The set of all positive real numbers is a group under multiplication.

   (c) The set $\mathbb{R}^{\geq 0} = \{r \in \mathbb{R} \mid r \geq 0\}$ with the operation $a \star b = \max(a, b)$ is a group.

   (d) The set $\mathbb{R}^{\times} = \{r \in \mathbb{R} \mid r \neq 0\}$ with the operation $a \star b = \frac{a}{b}$ is a group.

   (e) The set $\mathbb{R} - \{-1\}$ with the operation $a \star b = a + b + ab$ is a group.

   (f) If A and B are groups with operations $\star_A$ and $\star_B$, then $A \times B$ with operation

   $$(a_1, b_1) \star (a_2, b_2) = (a_1 \star_A a_2, b_1 \star_B b_2)$$

   is a group.

   (g) Every infinite group contains an element of infinite order.

2. (Based on a problem from the book, p. 147, # 2.47) What is the largest order of an element in S_n for $n = 3, 4, 5, 6, 7, 8, 9, 14,$ & 16?

3. (From the book, p. 147, # 2.46) Let G be a finite group with an even number of elements. Prove that G must contain an odd number of elements of order 2. In particular, G must contain an element of order 2.

4. Let G be a group. Prove that G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

5. Let G be a finite set equipped with an associative, binary operation given by a table in which each element of the set G appears exactly once in each row and column. Prove that G is a group. How do you recognize the identity element? How do you recognize the inverse of an element?

6. (From the book, p. 147, # 2.42) Let $G = GL_2(\mathbb{Q})$, and let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Show that $A^4 = \mathbb{1} = B^6$, but that $(AB)^n \neq \mathbb{1}$ for all $n > 0$. Conclude that AB can have infinite order even though both factors A and B have finite order.