Math 4320

Homework 4

Due FRIDAY 2/20/15

Name: ______________________________

Collaborators: ______________________________

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Please print out this page. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please **staple** *this cover sheet and your solutions together and hand in your homework in class.*

GRADE

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(For instructors's use only)

Exercises.

1. **True / False**. (From the book, p. 157, # 2.52 and p. 169, # 2.64)Determine whether the following statements are true or false. Justify your response with a proof or counterexample.

   (a) If G is a finite group, and m is a divisor of $|G|$, then G contains an element of order m.

   (b) The intersection of two cyclic subgroups of G is itself a cyclic subgroup.

   (c) Every proper subgroup of S_3 is cyclic.

   (d) Every proper subgroup of S_4 is cyclic.

   (e) The inclusion $\mathbb{Z} \to \mathbb{R}$ is a homomorphism of additive groups.

   (f) Any two finite groups of the same order are isomorphic.

   (g) If p is a prime, any two groups of order p are isomorphic.

2. (**Complex numbers**)

   (a) Show that the set $\mathbb{C}^\times$ of non-zero complex numbers $x + iy$ is a group under the operation of multiplication

   $$(x_1 + iy_1) * (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_2y_1 + x_1y_2).$$

   (b) We can also describe a non-zero complex number by its radius and angle:

   $$x + iy = r\, e^{i\theta} = r\,(\cos(\theta) + i\sin(\theta)),$$

   where $r = \sqrt{x^2 + y^2}$ and $0 \le \theta < 2\pi$ is the angle, measured in radians, counter-clockwise from the x-axis to the vector $(x, y) \in \mathbb{R}^2$. With this description, using the double angle formuæ, one can prove that the group operation is

   $$r_1\, e^{i\theta_1} * r_2\, e^{i\theta_2} = r_1r_2\, e^{i(\theta_1+\theta_2)}.$$

   (**You do not need to verify this**, but you may want to use it for the following.)

   Describe (or draw a picture of!) the various cyclic subgroups $\langle r\, e^{i\theta} \rangle$.

3. (From the book, p. 158, # 2.54)

   (a) Define the **special linear group** by

   $$\mathrm{SL}_2(\mathbb{R}) = \{A \in \mathrm{GL}_2(\mathbb{R}) \mid \det(A) = 1\}.$$

   Prove that $\mathrm{SL}_2(\mathbb{R})$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

   (b) A matrix $A \in \mathrm{GL}_2(\mathbb{R})$ is **stochastic** if for each row, the entries add up to 1. Show that the set of stochastic matrices $\Sigma_2(\mathbb{R})$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$.

4. (From the book, p. 158, # 2.57) If H and K are subgroups of a group G, and if $|H|$ and $|K|$ are relatively prime, prove that $H \cap K = \{\mathbb{1}\}$.

5. (From the book, p. 158, # 2.58) Prove that every infinite group contains infinitely many subgroups.

6. (From the book, p. 168, # 2.65) If there is a bijection $f : X \to Y$, prove that there is an isomorphism $\phi : S_X \to S_Y$.

7. (From the book, p. 171, # 2.71) Prove that the dihedral group of order 6 is isomorphic to S_3.