Math 4320

Homework 5

Due 2/25/15

Name: ______________________________

Collaborators: ______________________________

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Please print out this page. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please **staple** *this cover sheet and your solutions together and hand in your homework in class.*

GRADE
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(For instructors's use only)

Exercises.

1. **True / False**. (From the book, p. 169, # 2.64 and p. 190, # 2.95) Determine whether the following statements are true or false. Justify your response with a proof or counterexample.

   (a) The function $\phi : (GL_2(\mathbb{R}), \cdot) \to (\mathbb{R}^\times, \cdot)$ defined by

   $$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab$$

   is a homomorphism.

   (b) The function $\psi : (\mathbb{R}, +) \to (M_{2\times 2}(\mathbb{R}), +)$ defined by

   $$\psi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

   is a homomorphism.

   (c) There is a homomorphism $\mathbb{Z} \to \mathbb{I}_m$ which sends $a \in \mathbb{Z}$ to its equivalence class $[a] \in \mathbb{I}_m$.

   (d) There is a homomorphism $\mathbb{I}_m \to \mathbb{Z}$ which sends $[a] \in \mathbb{I}_m$ to $a \in \mathbb{Z}$.

   (e) The inverse function of an isomorphism is an isomorphism.

   (f) For any group G with normal subgroup $K \lhd G$, there is a homomorphism $G \to G/K$ with kernel precisely K.

2. Prove that if G and H are isomorphic groups, and $K \leq G$, then H has a subgroup isomorphic to K.

3. (From the book, p. 170, # 2.84) Show that the center of $GL_2(\mathbb{R})$ is the set of *scalar matrices*

   $$\left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \,\middle|\, a \in \mathbb{R}^\times \right\}.$$

4. (From the book, p. 171, # 2.92) An **automorphism** of a group G is an isomorphism $G \to G$.

   (a) Prove that $\mathrm{Aut}(G)$, the set of all automorphisms of G, is a group under composition.

   (b) Prove that $\gamma : G \to \mathrm{Aut}(G)$ defined by $g \mapsto \gamma_g$ (conjugation by g) is a homomorphism.

   (c) Prove that $\ker(\gamma) = Z(G)$.

   (d) Prove that $\mathrm{im}(\gamma) \lhd \mathrm{Aut}(G)$.

5. (From the book, p. 190, # 2.96) Prove that $U(\mathbb{I}_9) \cong \mathbb{I}_6$ and $U(\mathbb{I}_{15}) \cong \mathbb{I}_4 \times \mathbb{I}_2$.

6. An element $[a] \in \mathbb{I}_m$ is **nilpotent** if $[a]^k = [0]$ for some $k > 0$. An element $[a] \in \mathbb{I}_m$ is **idempotent** if $[a]^2 = [a]$.

   (a) Find all the nilpotent elements in $\mathbb{I}_8$.

   (b) Find all the idempotent elements in $\mathbb{I}_6$ and $\mathbb{I}_{12}$.

   (c) Show that $\mathbb{I}_m$ has no non-zero nilpotent elements if and only if n has no factor that is a perfect square (except 1).

   (d) Show that $[0]$ and $[1]$ are the only idempotent elements in $\mathbb{I}_p$ if p is prime.

7. Let G be an abelian group, and consider the map $\phi_n : G \to G$ defined by $\phi_n(g) = g^n$.

   (a) Show that ϕ_n is a homomorphism for every $n \in \mathbb{Z}$.

   (b) Let $G = U(\mathbb{I}_{15})$. Compute the kernel and image of $\phi_2 : G \to G$.