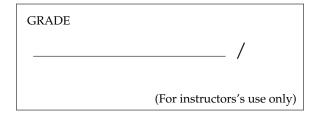
SS UNIV	Math 4320	Name:
	Homework 5	Collaborators:
	Due 2/25/15	

Please print out this page. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please **staple** this cover sheet and your solutions together and hand in your homework in class.



Exercises.

- 1. **True / False**. (From the book, p. 169, # 2.64 and p. 190, # 2.95) Determine whether the following statements are true or false. Justify your response with a proof or counterexample.
 - (a) The function $\phi : (GL_2(\mathbb{R}), \cdot) \to (\mathbb{R}^{\times}, \cdot)$ defined by

$$\Phi\left(\left[\begin{array}{cc}a&b\\c&d\end{array}\right]\right)=ab$$

is a homomorphism.

(b) The function $\psi : (\mathbb{R}, +) \to (M_{2 \times 2}(\mathbb{R}), +)$ defined by

$$\psi(\mathfrak{a}) = \left[\begin{array}{cc} 1 & \mathfrak{a} \\ 0 & 1 \end{array} \right]$$

is a homomorphism.

- (c) There is a homomorphism $\mathbb{Z} \to \mathbb{I}_m$ which sends $a \in \mathbb{Z}$ to its equivalence class $[a] \in \mathbb{I}_m$.
- (d) There is a homomorphism $\mathbb{I}_m \to \mathbb{Z}$ which sends $[a] \in \mathbb{I}_m$ to $a \in \mathbb{Z}$.
- (e) The inverse function of an isomorphism is an isomorphism.

(f) For any group G with normal subgroup K \lhd G, there is a homomorphism G \rightarrow G/K with kernel precisely K.

- 2. Prove that if G and H are isomorphic groups, and $K \leq G$, then H has a subgroup isomorphic to K.
- 3. (From the book, p. 170, # 2.84) Show that the center of $GL_2(\mathbb{R})$ is the set of *scalar matrices*

$$\left\{ \left[\begin{array}{cc} a & 0 \\ 0 & a \end{array} \right] \ \middle| \ a \in \mathbb{R}^{\times} \right\}.$$

HW5

- 4. (From the book, p. 171, # 2.92) An automorphism of a group G is an isomorphism G → G.
 (a) Prove that Aut(G), the set of all automorphisms of G, is a group under composition.
 - (b) Prove that $\gamma : G \to Aut(G)$ defined by $g \mapsto \gamma_g$ (conjugation by g) is a homomorphism.
 - (c) Prove that $ker(\gamma) = Z(G)$.
 - (d) Prove that $im(\gamma) \lhd Aut(G)$.
- 5. (From the book, p. 190, # 2.96) Prove that $U(\mathbb{I}_9) \cong \mathbb{I}_6$ and $U(\mathbb{I}_{15}) \cong \mathbb{I}_4 \times \mathbb{I}_2$.
- 6. An element $[a] \in \mathbb{I}_m$ is **nilpotent** if $[a]^k = [0]$ for some k > 0. An element $[a] \in \mathbb{I}_m$ is **idempotent** if $[a]^2 = [a]$.
 - (a) Find all the nilpotent elements in \mathbb{I}_8 .
 - (b) Find all the idempotent elements in \mathbb{I}_6 and \mathbb{I}_{12} .

(c) Show that \mathbb{I}_m has no non-zero nilpotent elements if and only if n has no factor that is a perfect square (except 1).

(d) Show that [0] and [1] are the only idempotent elements in \mathbb{I}_p if p is prime.

- 7. Let G be an abelian group, and consider the map $\phi_n : G \to G$ defined by $\phi_n(g) = g^n$.
 - (a) Show that ϕ_n is a homomorphism for every $n \in \mathbb{Z}$.
 - (b) Let $G = U(\mathbb{I}_{15})$. Compute the kernel and image of $\phi_2 : G \to G$.