Pseudo-Modularity and Iwasawa Theory

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The Ordinary Eigencurve

Fix a prime p ($p \geq 5$ later on) and a level $N \geq 1$ ($p \nmid N\varphi(N)$ later on).

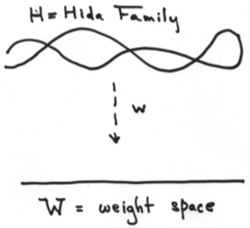
Hida Theory

Any ordinary (cuspidal) eigenform of weight k and level N

$$f(q) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots$$

lies in a p-adic family of (cuspidal) eigenforms with p-adically varying weight k. "Ordinary" means that a_p is a p-adic unit.

Picture credit: Barry Mazur



Weight space is $\mathrm{Spec}\,\Lambda[1/p]$, where $\Lambda = \mathbb{Z}_p[\![t]\!]$.

Think of a weight $k \in \mathbb{Z}_p$ as a map $\Lambda \to \mathbb{Z}_p$ sending t to $(1+p)^k - 1$.

The Ordinary Eigencurve

Example: Eisenstein series

The (classical) Eisenstein series of weight $k \in \mathbb{Z}_{\geq 2}$ and level $N = 1$:

$$E_k(q) = \zeta(k)/2 + \sum_{n \geq 1} q^n \sum_{d|n} d^k = \zeta(k)/2 + \sum_{n \geq 1} \sigma_n(k) q^n.$$

After a slight modification, the coefficients vary p-analytically in terms of k:

$$E(k,q) = \xi(k) + \sum_{n \geq 1} q^n \sum_{d|n,\, p \nmid d} d^k = \xi(k) + \sum_{n \geq 1} \sigma_n^{(p)}(k) q^n$$

Each $\sigma_n^{(p)}$ along with ξ are elements of $\Lambda = \mathbb{Z}_p[\![t]\!]$. The image of $\sigma_n^{(p)}$ under $t \mapsto (1+p)^k - 1$ is $\sigma_n^{(p)}(k)$.

The Ordinary Eigencurve

The Hida Hecke Algebras

Write $\mathfrak{H}$ for the Eisenstein component of the ring of Hecke operators acting on all p-adic families of ordinary eigenforms. Write $\mathfrak{h}$ for the quotient of $\mathfrak{H}$ arising from the Hecke action on *cuspidal* ordinary eigenforms.

Hida theory $\implies$ $\mathfrak{H}$ and $\mathfrak{h}$ are complete local *finite and flat* Λ-algebras.

Think: an ordinary modular form with coefficients in a ring A corresponds to a map $\mathfrak{H} \to A$.

Example

The ordinary Eisenstein series $E = E(k, q)$ has coefficients in Λ, so there exists a map $\mathfrak{H} \twoheadrightarrow \Lambda$ corresponding to it.

The Ordinary Eigencurve

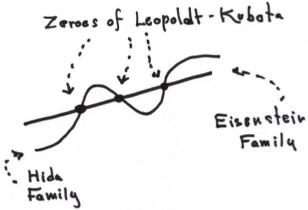
Definition

The **ordinary eigencurve** *is* $\mathrm{Spec}\ \mathfrak{H}[1/p]$*. The* **cuspidal locus** *and* **Eisenstein locus** *are the closed subschemes*

$$\mathrm{Spec}\ \mathfrak{h}[1/p] \subset \mathrm{Spec}\ \mathfrak{H}[1/p], \qquad \mathrm{Spec}\ \Lambda[1/p] \subset \mathrm{Spec}\ \mathfrak{H}[1/p],$$

respectively.

The map $\mathfrak{H} \twoheadrightarrow \Lambda$ cutting out the Eisenstein locus corresponds to the Eisenstein series.



These two loci intersect whenever ξ has a zero at some weight.

Picture credit: Barry Mazur

Eisenstein Intersection Points

Question

What is the nature of these crossings of Eisenstein and cuspidal families?

Conjecture (Preston Wake)

At an intersection point, the local rings $\mathfrak{H}_\mathfrak{p}$ and $\mathfrak{h}_\mathfrak{p}$ of $\mathfrak{H}[1/p]$ and $\mathfrak{h}[1/p]$ at this point are Gorenstein.

This conjecture for all intersection points is called "weak Gorensteinness."

In other words, the singularity at $\mathfrak{p}$ should be not too bad. For example, if it is complete intersection (i.e. has the form $K[\![x_1, \ldots, x_n]\!]/(f_1, \ldots, f_m)$ with Krull dimension equal to $n - m$), then it is Gorenstein.

Motivation for the Weak Gorensteinness Conjecture

Motivation from the integral case:

- The ordinary part of the cohomology of the tower of modular curves $X_0(p^n N)$ is a $\mathfrak{h}$-module H with a linear Galois action. This module is generically free of rank 2, but is free of rank 2 if and only if $\mathfrak{h}$ is Gorenstein.
- Gorensteinness is sometimes known for $\mathfrak{H}$ and $\mathfrak{h}$ (integrally), and has close relations with Iwasawa towers of class groups "X":
  - Wake: If $\mathfrak{H}$ is Gorenstein, then $X^+ = 0$. ($X^+ = 0$ is Vandiver's conjecture when $N = 1$; also need $X^- \neq 0$)
  - Wake: If Sharifi's conjecture is true, then $X^+ = 0 \implies \mathfrak{H}$ is Gorenstein.
    - Sharifi's conjecture proposes an isomorphism between X^- and a certain quotient H^-/IH^- of H, refining the Iwasawa main conjecture.
  - Wake's Corollary: $\mathfrak{H}$ is not always Gorenstein. (Find $X^+ \neq 0$.)

Motivation for the Weak Gorensteinness Conjecture

We have seen that $\mathfrak{H}$ is not always Gorenstein, but perhaps $\mathfrak{H}[1/p]$ and $\mathfrak{h}[1/p]$ are still Gorenstein along the Eisenstein locus; this is Wake's "weak Gorensteinness" conjecture.

- Similar consequences about class groups and representation theory are true.
- In particular, $H_{\mathfrak{p}} := H \otimes_{\mathfrak{h}} \mathfrak{h}_{\mathfrak{p}}$ is a "true 2-dimensional representation" over $\mathfrak{h}_{\mathfrak{p}}$ if and only if $\mathfrak{h}_p$ is Gorenstein.

Theorem (Wake)

$\mathfrak{H}$ is weakly Gorenstein if and only if both of the following are true:

1. *Sharifi's conjecture ($X^- \xrightarrow{\sim} H^-/IH^-$) is "weakly" true, i.e. the kernel and cokernel are finite cardinality, and*
2. *A certain quotient of X^+ is finite cardinality.*

Main Result on the Eigencurve

X^+ is expected to be finite cardinaltiy – this is Greenberg's pseudo-nullity conjecture. So Wake's theorem implies that weak Gorensteinness follows from weak Sharifi's conjecture and Greenberg's conjecture.

Our main theorem proves that Greenberg's pseudo-nullity conjecture is sufficient to imply weak Gorensteinness (and thereby also to imply weak Sharifi's conjecture).

Theorem (Wake-W.E.)

- *Assuming Greenberg's conjecture that X^+ is finite cardinality, $\mathfrak{H}$ and $\mathfrak{h}$ are weakly Gorenstein. Moreover, $\mathfrak{H}_\mathfrak{p}$, $\mathfrak{h}_\mathfrak{p}$ are compl. intersection.*
- *Moreover, ξ has at most simple zeros if and only if $\mathfrak{h}_\mathfrak{p}$ are DVRs, i.e. the cuspidal locus is smooth at each Eisenstein intersection point.*

Note: Simple zeros implies Greenberg's conjecture, but the converse is not true.

Method of Proof: Pseudo-Modularity

Although $G_{\mathbb{Q}} \to \mathrm{End}_{\mathfrak{h}_\mathfrak{p}}(H_\mathfrak{p})$ may not be a true representation (until we prove that $\mathfrak{h}_\mathfrak{p}$ is Gorenstein, $H_\mathfrak{p}$ may not be free), it still induces a trace and determinant function.

This data is a 2-dimensional *pseudorepresentation* – i.e. the data of a trace and determinant function on $G_{\mathbb{Q}}$ satisfying coherence conditions expected of such functions coming from a representation.

Theorem (Many people; Chenevier)

There is a universal deformation ring R for the residual Galois pseudorepresentation on $H_\mathfrak{p}/\mathfrak{p}H_\mathfrak{p}$.

There is then a surjection $R \twoheadrightarrow \mathfrak{h}_\mathfrak{p}$ induced by the $G_{\mathbb{Q}}$ action on $H_\mathfrak{p}$; we also produce $R \twoheadrightarrow \mathfrak{H}_\mathfrak{p}$.

We want to use a refinement of this map to prove $\mathfrak{H}_\mathfrak{p}, \mathfrak{h}_\mathfrak{p}$ to be Gorenstein.

Method of Proof: Pseudo-Modularity

1. Construct a "universal **ordinary pseudodeformation** ring" R^{ord}, a quotient of R, and verify that there is a factorization $R^{\mathrm{ord}} \twoheadrightarrow \mathfrak{H}_\mathfrak{p}$.
2. Use the assumption of Greenberg's conjecture to control certain Galois cohomology groups.
3. Use a result of Bellaïche that characterizes the tangent space of R^{ord} in terms of Galois cohomology, showing that it is 2-dimensional.
4. Hence the cotangent space $\mathfrak{p}/\mathfrak{p}^2$ of $\mathfrak{H}_\mathfrak{p}$ is at most 2-dimensional; because $\mathfrak{H}_\mathfrak{p}$ is Krull 1-dimensional, it is complete intersection, hence Gorenstein. The same is true for $\mathfrak{h}_\mathfrak{p}$, proving the theorem.
5. Using the same commutative algebra Wiles used in his Fermat paper:

Theorem (Wake-W.E.)

The Galois action on the cohomology of modular curves induces an isomorphism $R^{\mathrm{ord}} \xrightarrow{\sim} \mathfrak{H}_\mathfrak{p}$. "$R = \mathbb{T}$"

Key Method: Ordinary Pseudorepresentations

The notion of "ordinary pseudorepresentation of $G_{\mathbb{Q}}$" is subtle:

- "Ordinary" for a 2-dimensional representation V of $G_{\mathbb{Q}}$ means that the restriction to a decomposition group G_p at p is reducible, with a quotient that is unramified.
- However, pseudorepresentations are characteristic polynomial coefficients, and characteristic polynomial coefficients don't see which direction an extension goes!
- Nevertheless, an irreducible pseudorepresentation of $G_{\mathbb{Q}}$ can see extensions between factors of the restriction to G_p!

Definition (Ordinary pseudorepresentation, first attempt)

An ordinary pseudorepresentation is a pseudorepresentation induced by some ordinary representation.

But this does not work... not every pseudorepresentation comes from a representation.

Key Method: Ordinary Pseudorepresentations

Solution

Expand the category of representations to include *Cayley-Hamilton representations* of $G_{\mathbb{Q}}$, a notion due (in this form) to Chenevier and Bellaïche-Chenevier: a C-H representation of $G_{\mathbb{Q}}$ is

- an A-algebra R,
- a pseudorepresentation $D : R \to A$, and
- a homomorphism $\rho : G_{\mathbb{Q}} \to R^{\times}$.

Every pseudorepresentation does come from a Cayley-Hamtilon representation via the pseudorepresentation $D \circ \rho$ of $G_{\mathbb{Q}}$.

Definition (Ordinary pseudorepresentation, successful attempt)

An ordinary pseudorepresentation is a pseudorepresentation induced by some ordinary Cayley-Hamilton representation.

Of course, "ordinary C-H representation" needs to be defined.

Ordinary Cayley-Hamilton Representations

Any Cayley-Hamtilon representation $\rho : G_{\mathbb{Q}} \to R$ of $G_{\mathbb{Q}}$ over A with residual pseudorepresentation identical to that of $H_{\mathfrak{p}}/\mathfrak{p}H_{\mathfrak{p}}$ will have two primitive orthogonal idempotents, giving R and ρ matrix decomposition

$$R \simeq \begin{pmatrix} A & B \\ C & A \end{pmatrix}, \qquad \rho = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} \\ \rho_{2,1} & \rho_{2,2} \end{pmatrix}$$

where B, C are A-modules with a multiplication law $B \otimes_A C \to A$ satisfying those identities necessary to make R an associative algebra.

- This is a "generalized matrix algebra" of Bellaïche-Chenevier.

Definition (Ordinary Cayley-Hamilton Representation)

Call a Cayley-Hamilton representation ordinary when $\rho_{1,2}(G_p) = 0$ *and* $\rho_{1,1}(I_p) = 1$.

Non-commutative Modularity

There is actually a universal such Cayley-Hamilton representation E^{ord} with center R^{ord}, we we can actually prove a non-commutative strengthening of our pseudo-modularity theorem:

Theorem (Wake-W.E.)

Assuming Greenberg's conjecture as before, the canonical map $E^{\mathrm{ord}} \otimes_{R^{\mathrm{ord}}} \mathfrak{h}_{\mathfrak{p}} \to \mathrm{End}_{\mathfrak{h}_{\mathfrak{p}}}(H_{\mathfrak{p}})$ is injective, i.e. every ordinary cuspidal Cayley-Hamilton representation is realized in the ordinary cohomology of modular curves.

A similar statement should be true for E^{ord} itself, but is more complicated to state.