THE ERDŐS–GALLAI CRITERION AND SYMMETRIC FUNCTIONS

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Summary Erdős and Gallai [Erd-Gal] gave a necessary and sufficient condition for an integer sequence to be the degree sequence of a simple graph. We present a proof of this criterion based on the theory of symmetric functions.

1. Introduction

Let G be a simple graph (i.e., G has no loops or multiple edges) on n vertices. The *degree*, $\deg(v)$, of a vertex v in G is the number of edges incident with v. Assume that the vertices $v_1, v_2, \ldots, v_n$ of G are listed in non-increasing order of their degrees and let $d_i = \deg(v_i)$ The sequence $d(G) = (d_1, d_2, \ldots, d_n)$ is called the *degree sequence* of G. There are many sequences that are not degree sequences of graphs, for example $\mathbf{d} = (3, 3, 3, 1)$. So it is natural to ask for a necessary and sufficient condition that a non-increasing integer sequence is the degree sequence of a simple graph. The answer is given by the following theorem which is due to Erdős and Gallai [Erd-Gal] and is called the Erdős–Gallai criterion.

1.1 Theorem. *Let $d_1 \geq d_2 \geq \cdots \geq d_n$ be a sequence of positive integers. Then $(d_1, d_2, \ldots, d_n)$ is the degree sequence of a simple graph if and only if $(d_1 + d_2 + \cdots + d_n)$ is even and $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$ for $k = 1, 2, \ldots, n$.* $\square$

Below we show how the problem can be approached using the theory of symmetric functions. All undefined terminology can be found in [Mac].

2. The symmetric functions approach

Let us associate with each edge $v_i v_j$ of G the quadratic monomial $x_i x_j$. Then we associate with G the product of the monomials corresponding to the edges of G. This gives a map ϕ from the set of simple graphs with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ to the set of monomials of even degree in the indeterminates $x_1, x_2, \ldots, x_n$. It is clear that we can read off the degree sequence of a graph G from $\phi(G)$:

$$d(G) = (d_1, d_2, \ldots, d_n) \iff \phi(G) = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}.$$

Consider the product

$$\prod_{1 \leq i < j \leq n} (1 + x_i x_j) = \sum_{\{\epsilon_{i,j} \in \{0,1\} \ | \ 1 \leq i < j \leq n\}} \prod_{1 \leq i < j \leq n} (x_i x_j)^{\epsilon_{i,j}}.$$

Each term $\prod_{1\le i<j\le n}(x_i x_j)^{\epsilon_{i,j}}$ corresponds to a simple graph on V with edge set $\{v_i v_j \mid \epsilon_{i,j}=1\}$ and vice versa. This shows that $(d_1, d_2, \ldots, d_n)$ is a degree sequence of a simple graph if and only if the coefficient of $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ in $\prod_{1\le i<j\le n}(1+x_i x_j)$ is different from 0.

The following symmetric function identity is due to Littlewood [Lit, p. 238] and has been given a combinatorial proof by Burge [Bur].

2.1 Lemma. $\prod_{1\le i<j}(1+x_i x_j) = \sum_\lambda s_\lambda$, *where* λ *runs over all partitions of the form* $(\alpha_1 - 1, \alpha_2 - 1, \ldots, \alpha_p - 1 \mid \alpha_1, \alpha_2, \ldots, \alpha_p)$ *in Frobenius notation and* s_λ *is the Schur function indexed by* λ. $\square$

This shows that $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is a degree sequence of a simple graph if and only if there exists a partition λ as above such that the coefficient of $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ in s_λ is different from 0. The last condition is equivalent to $\mathbf{d} \le \lambda$, where " $\le$ " is the dominance order for partitions.

Now we will show that this is equivalent to the Erdős–Gallai criterion.

2.2 Definition. Let d be a partition. The *Durfee square* of d is the maximal square which is contained in the Young diagram of d and whose upper-left corner coincides with the upper-left corner of the Young diagram of d. We will denote the size of the Durfee square of d by $s = s(d)$. Alternatively, s is the largest number such that $d_s \ge s$.

Let $l(\mathbf{d})$ be the length of $\mathbf{d}$.

2.3 Lemma. *Let* d *be a partition. Then the following two conditions are equivalent:*

(1) $\sum_{i=1}^k d_i \le k(k-1) + \sum_{i>k} \min(d_i, k)$ for $1 \le k \le l(d)$

(2) $\sum_{i=1}^k d_i \le k(k-1) + \sum_{i>k} \min(d_i, k)$ for $1 \le k \le s(d)$

Proof. It is clear that (1) implies (2). Now assume that $\mathbf{d}$ satisfies (2). Let $s = s(\mathbf{d})$. We want to prove that if $k > s$, then

$$\sum_{i=1}^k d_i \le k(k-1) + \sum_{i>k} \min(d_i, k).$$

By definition $d_i < s$ if $i > s$, so the above inequality is equivalent to

$$\sum_{i=1}^k d_i \le k(k-1) + \sum_{i>k} d_i.$$

Since by assumption

$$\sum_{i=1}^s d_i \le s(s-1) + \sum_{i>s} \min(d_i, s) = s(s-1) + \sum_{i>s} d_i,$$

it will be enough to prove that

$$(s(s-1)+\sum_{i>s} d_i)+\sum_{i=s+1}^{k} d_i \leq k(k-1)+\sum_{i>k} d_i,$$

i.e.,

$$s(s-1)+2\sum_{i=s+1}^{k} d_i \leq k(k-1).$$

But for $i \geq s+1$ we have that $d_i \leq d_{s+1} < s+1$, hence $d_i \leq s$. Thus it will be enough to show that

$$s(s-1)+2(k-s)s \leq k(k-1),$$

which can be transformed to

$$(k-s)(k-s-1) \geq 0.$$

The last inequality follows from the fact that $k \geq s+1$. □

2.4 Definition.

(1) We say that a partition $\mathbf{d}$ satisfies the *Erdős–Gallai condition* if it satisfies either (1) or (2) in Lemma 2.3. (Note that (1) in Lemma 2.3 is the condition that appears in the Erdős–Gallai criterion.) The set of all partitions of N that satisfy the Erdős–Gallai condition is denoted by $\mathrm{EG}(N)$.

(2) We say that $\mathbf{d}$ satisfies the *"closed" Erdős–Gallai condition* if $\sum_{i=1}^{k} d_i = k(k-1)+\sum_{i>k}\min(d_i,k)$ for $1 \leq k \leq s(d)$. The set of all partitions of N that satisfy the closed Erdős–Gallai condition is denoted by $\widetilde{\mathrm{EG}}(N)$.

(3) The *Erdős–Gallai sequence* of $\mathbf{d}$ is the sequence $\alpha = \alpha(\mathbf{d}) = (\alpha_1, \alpha_2, \dots, \alpha_s)$, where $\alpha_i = 1 + d_i - d'_i$ and $s = s(d)$ is the size of the Durfee square of d.

Note that $\widetilde{\mathrm{EG}}(N) \subseteq \mathrm{EG}(N)$.

We will extend the notion of dominance order for partitions to arbitrary sequences of real numbers. Namely, if $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$ are two sequences of real numbers, then we say that $\mathbf{a} \leq \mathbf{b}$ if

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$$

for all $k \geq 1$. (As usual, if $\mathbf{c} = (c_1, c_2, \dots, c_n)$ is a finite sequence, then we let $c_i = 0$ for $i > n$.) We will also denote by $\mathbf{0}$ the zero sequence $(0, 0, \dots)$.

2.5 Lemma.

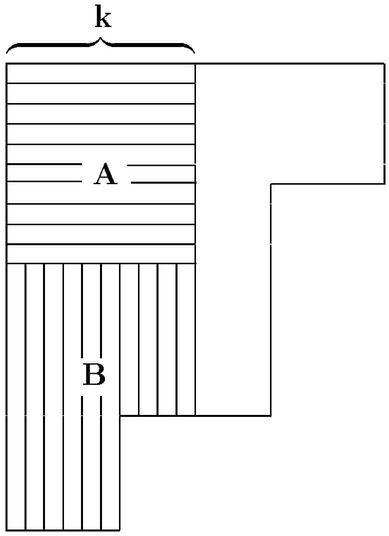
(1) $\mathbf{d} \in \mathrm{EG}(N) \iff \alpha \leq \mathbf{0}$.

(2) $\mathbf{d} \in \widetilde{\mathrm{EG}}(N) \iff \alpha = \mathbf{0} \iff \mathbf{d} = (a_1 - 1, a_2 - 1, \dots, a_s - 1 \mid a_1, a_2, \dots, a_s)$ for some $a_1, a_2, \dots, a_s$. *In particular,* $\widetilde{\mathrm{EG}}(N) \neq \emptyset \iff 2/N$.

Proof. First we will show that for any $k \leq s$,

$$\sum_{i=1}^{k} d'_i = k^2 + \sum_{i>k} \min(d_i, k)$$

This is best done by drawing a picture:



This picture represents schematically the Young diagram of $\mathbf{d}$. In it A is the intersection of the first k rows with the first k columns and B is the intersection of the $k+1, k+2, \ldots$ rows with the first k columns of the Young diagram of d. Note that the condition $k \leq s$ guarantees that A is a square of size k.

Then, assuming that the cells of the Young diagram of d have unit size, we see that B has area $\sum_{i>k} \min(d_i, k)$. On the other hand, $A \cup B$ is the union of the first k columns of the Young diagram of d, so

$$\sum_{i=1}^{k} d'_i = k^2 + \sum_{i>k} \min(d_i, k).$$

Therefore for $k \leq s$ we have that

$$(*) \qquad \begin{aligned} \sum_{i=1}^{k} \alpha_i &= \sum_{i=1}^{k} (1 + d_i - d'_i) \\ &= k + \sum_{i=1}^{k} d_i - \sum_{i=1}^{k} d'_i \\ &= k + \sum_{i=1}^{k} d_i - [k^2 + \sum_{i>k} \min(d_i, k)] \\ &= \sum_{i=1}^{k} d_i - [k(k-1) + \sum_{i>k} \min(d_i, k)]. \end{aligned}$$

Thus

$$\begin{aligned}\alpha \le \mathbf{0} &\iff \sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i>k} \min(d_i, k) \ \mathit{for}\ k \le s \\ &\iff \mathbf{d} \in \mathrm{EG}(N).\end{aligned}$$

This proves (1). Since

$$\alpha = \mathbf{0} \iff \sum_{i=1}^{k} \alpha_i = 0 \ \mathit{for}\ k \le s,$$

we get from $(*)$ that

$$\alpha = \mathbf{0} \iff \sum_{i=1}^{k} d_i = k(k-1) + \sum_{i>k} \min(d_i, k)$$

which proves the first equivalence in (2). Also, $\alpha = \mathbf{0}$ is equivalent to $d_i = d_i' - 1$ for $i \le s$, which proves the second equivalence in (2).

To prove the last assertion in (2), note that if

$$\widetilde{\mathrm{EG}}(N) \ni \mathbf{d} = (a_1 - 1, \ldots, a_s - 1 \mid a_1, \ldots, a_s),$$

then

$$N = 2(a_1 + a_2 + \cdots + a_s),$$

so N is even. On the other hand, if N is even, then

$$\widetilde{\mathrm{EG}}(N) \ni (\frac{N}{2} - 1 \mid \frac{N}{2}). \quad \square$$

Lemma 2.5 (2) shows in particular that the Erdős–Gallai criterion is equivalent to the following:

2.6 Theorem. *Let* $\mathbf{d}$ *be a partition of* N. *Then there exists a* $\tilde{\mathbf{d}} \in \widetilde{\mathrm{EG}}(N)$ *such that* $\mathbf{d} \le \tilde{\mathbf{d}}$ *if and only if* N *is even and* $\mathbf{d} \in \mathrm{EG}(N)$.

Proof. The "only if" part follows from Lemma 2.5 (1) and Lemma 2.7 below. Since there are only finitely many partitions of N, the "if" part follows from Proposition 2.8 below. $\square$

2.7 Lemma. *If* $\mathbf{d}, \mathbf{e} \in \mathrm{P}_N$ *and* $\mathbf{d} \le \mathbf{e}$, *then* $\alpha(\mathbf{d}) \le \alpha(\mathbf{e})$.

Proof. It is well-known that $\mathbf{d} \le \mathbf{e}$ implies $\mathbf{d}' \ge \mathbf{e}'$. Thus for any k we have that

$$\begin{aligned}\sum_{i=1}^{k} \alpha_i(\mathbf{d}) &= k + \sum_{i=1}^{k} d_i - \sum_{i=1}^{k} d_i' \\ &\le k + \sum_{i=1}^{k} e_i - \sum_{i=1}^{k} e_i' \\ &= \sum_{i=1}^{k} \alpha_i(\mathbf{e}),\end{aligned}$$

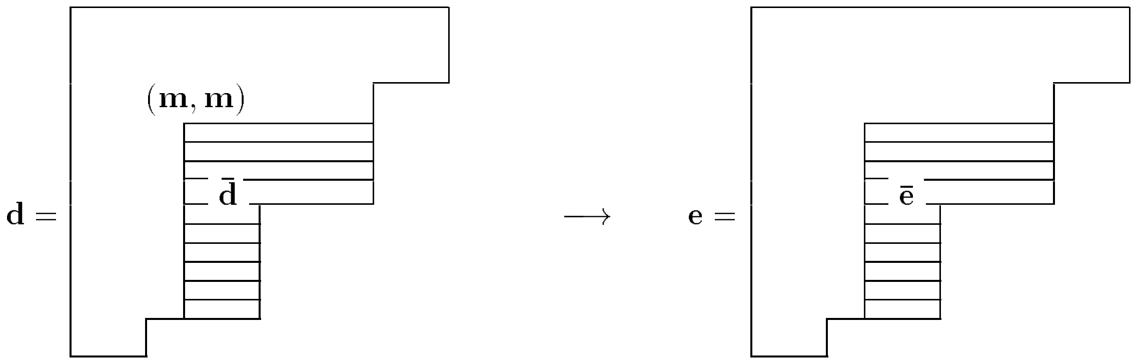
i.e.,

$$\alpha(\mathbf{d}) \le \alpha(\mathbf{e}). \quad \square$$

2.8 Proposition. *If* $\mathbf{d} \in \mathrm{EG}(N) - \widetilde{\mathrm{EG}}(N)$ *for some even* N*, then there exists an* $\mathbf{e} \in \mathrm{EG}(N)$ *such that* $\mathbf{d} < \mathbf{e}$*. Moreover, we can construct* $\mathbf{e}$ *in such a way that* $s(\mathbf{e}) = s(\mathbf{d}) = s$*.*

Note. The fact that $\mathbf{e}$ can be constructed such that $s(\mathbf{e}) = s(\mathbf{d})$ is not needed for the proof of the Erdős–Gallai criterion. It is however an interesting fact which requires very little additional effort to prove, so we include it here.

Proof of Proposition 2.8. Let $s = s(d)$ be the size of the Durfee square of $\mathbf{d}$. Let m be the smallest number in $\{1, 2, \dots, s\}$ such that $\sum_{i=1}^{m} d_i < m(m-1) + \sum_{i>m} \min(d_i, m)$. The condition that $\mathbf{d} \notin \widetilde{\mathrm{EG}}(N)$ ensures that m exists. Then by induction on k we see that $d'_k = d_k + 1$ for $k = 1, 2, \dots, m-1$ and $d'_m > d_m + 1$. Let $\bar{\mathbf{d}} = (d_m - m + 1, d_{m+1} - m + 1, \dots)$. It is easy to see that $\bar{\mathbf{d}} \in \mathrm{EG}(\bar{N}) - \widetilde{\mathrm{EG}}(\bar{N})$ for some even $\bar{N}$. We will construct a partition $\bar{\mathbf{e}} \in \mathrm{EG}(\bar{N})$ such that $\bar{\mathbf{d}} < \bar{\mathbf{e}}$, the first part of $\bar{\mathbf{e}}$ is at most $d_m - m + 2$, and the first part of the conjugate of $\bar{\mathbf{e}}$ is at most $d'_m - m + 1$. In terms of Young diagrams this means that the length of the first row of $\bar{\mathbf{e}}$ is at most $d_m - m + 2$ and the length of the first column of $\bar{\mathbf{e}}$ is at most $d'_m - m + 1$. Since $d_m < d'_m - 1 \le d'_{m-1} - 1 = d_{m-1}$ we have the inequality $d_m - m + 2 \le d_{m-1} - m + 1$, so we can "glue" the Young diagrams of $\mathbf{d} - \bar{\mathbf{d}}$ and $\bar{\mathbf{e}}$ to obtain a Young diagram of a partition $\mathbf{e} \in \mathrm{EG}(N)$ such that $\mathbf{d} < \mathbf{e}$. Schematically:



Construction of $\bar{\mathbf{e}}$:

Without loss of generality we can assume that $\mathbf{d} = \bar{\mathbf{d}}$, so we have the inequality $d'_1 \ge d_1 + 2$. Then we want to construct a partition $\mathbf{e} \in \mathrm{EG}(N)$ which satisfies the following conditions:

$$(**) \qquad \begin{aligned} &\mathbf{d} < \mathbf{e} \\ &e_1 \le d_1 + 1 \end{aligned}$$

We will consider three cases:

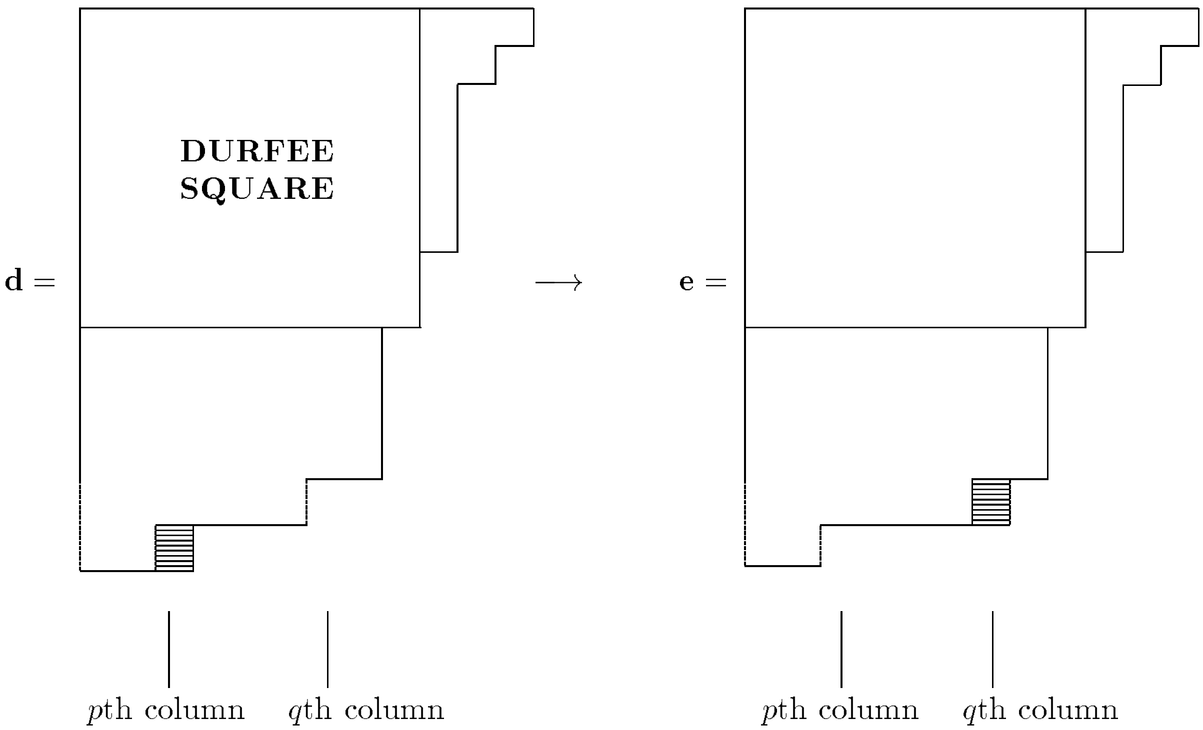
Case 1. $d'_1 - d'_s \geq 2$

Let $p = \max\{i \mid d'_1 = d'_i\}$ and $q = \min\{i \mid d'_1 - d'_i \geq 2\}$. It is clear that p and q exist and that $1 \leq p < q \leq s$. Define $\mathbf{e}$ as follows:

$$\begin{aligned} e'_i &= d'_i \text{ for } i \neq p, q \\ e'_p &= d'_p - 1 \\ e'_q &= d'_q + 1 \end{aligned}$$

Since $q \leq s$, we have that $e_i = d_i$ for $i \leq s$. Then for $i \leq q-1$ we have that $d'_1 - d'_i \leq 1$, so there are inequalities $d'_i \geq d'_1 - 1 \geq (d_1+2) - 1 = d_1 + 1 \geq d_i + 1$. Therefore $\alpha_i(d) = 1 + d_i - d'_i \leq 0$ for $i \leq q-1$. Moreover, $d'_p = d'_1 \geq d_1 + 2 \geq d_p + 2$. This implies that $e'_p = d'_p - 1 \geq d_p + 1 = e_p + 1$, i.e., $\alpha_p(\mathbf{e}) = 1 + e_p - e'_p \leq 0$. Since $e'_i = d'_i$ and $e_i = d_i$ for $i \leq q-1$, $i \neq p$ it follows that $\alpha_i(\mathbf{e}) \leq 0$ for any $i \leq q-1$. In particular, $\sum_{i=1}^k \alpha_i(\mathbf{e}) \leq 0$ for $k \leq q-1$. From the equality $\alpha_p(\mathbf{e}) + \alpha_q(\mathbf{e}) = \alpha_p(\mathbf{d}) + \alpha_q(\mathbf{d})$ and $\alpha_i(\mathbf{e}) = \alpha_i(\mathbf{d})$ for $i \leq s$, $i \neq p, q$, it follows that $\sum_{i=1}^k \alpha_i(\mathbf{e}) = \sum_{i=1}^k \alpha_i(\mathbf{d}) \leq 0$ for $q \leq k \leq s$. Therefore $\alpha(\mathbf{e}) \leq \mathbf{0}$, so from Lemma 2.5 (1) it follows that $\mathbf{e} \in \mathrm{EG}(N)$. It is also easy to see that $\mathbf{e}$ satisfies $(**)$ and that $s(\mathbf{e}) = s$.

Schematically:



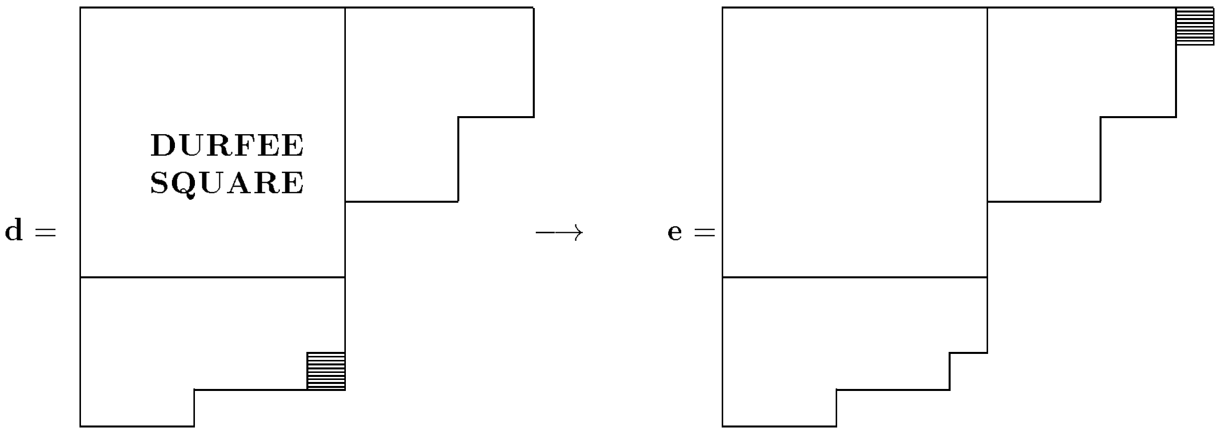
Case 2. $d'_1 - d'_s \leq 1$ and $s \geq 2$

Let

$$e_1 = d_1 + 1, e_i = d_i \text{ for } 2 \le i \le s,$$
$$e'_s = d'_s - 1, \text{ and } e'_i = d'_i \text{ for } i \le s-1.$$

Such $\mathbf{e}$ exists because $d'_s \ge d'_1 - 1 \ge d_1 + 1 \ge s+1$. In particular, $s(\mathbf{e}) = s(\mathbf{d}) = s$.

Schematically:



It is clear that $\mathbf{d} < \mathbf{e}$. We will prove that $\mathbf{e} \in \mathrm{EG}(N)$. We have that $e_1 = d_1 + 1$ and $e'_1 = d'_1$, so $\alpha_1(\mathbf{e}) = 1 + e_1 - e'_1 = 2 + d_1 - d'_1 \le 0$. If $2 \le i \le s-1$, then $e_i = d_i$ and $e'_i = d'_i$, so $\alpha_i(\mathbf{e}) = \alpha_i(\mathbf{d})$. Moreover, $d'_i \ge d'_s \ge d'_1 - 1 \ge d_1 + 1 \ge d_i + 1$, so $\alpha_i(\mathbf{e}) = \alpha_i(\mathbf{d}) = 1 + d_i - d'_i \le 0$.

If $i = s$, then $e'_s = d'_s - 1$ and $e_s = d_s$, so we obtain that $\alpha_s(\mathbf{e}) = 1 + e_s - e'_s = 2 + d_s - d'_s$. But as we showed above, there are inequalities $d'_s \ge d_1 + 1$ and $d_1 \ge d_s$, so $d'_s \ge d_s + 1$, hence $\alpha_s(\mathbf{e}) \le 1$. Thus we see that $\sum_{i=1}^{k} \alpha_i(\mathbf{e}) \le 0$ for $1 \le k \le s-1$ and $\sum_{i=1}^{s} \alpha_i(\mathbf{e}) \le 1$. But it is not hard to see that $\sum_{i=1}^{s} \alpha_i(\mathbf{e})$ is even, so $\sum_{i=1}^{s} \alpha_i(\mathbf{e}) \le 0$. This shows that $\mathbf{e} \in \mathrm{EG}(N)$. It also follows directly from the construction of $\mathbf{e}$ that $\mathbf{e}$ satisfies $(**)$.

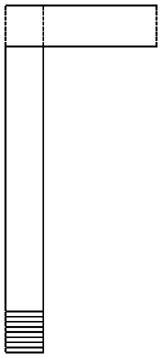
Case 3. $s = 1$, i.e., $\mathbf{d}$ is a hook

This is the easiest of the three cases that we consider. Since $s = 1$, we have that $N = d_1 + d'_1 - 1$. But N is even and $d'_1 \ge d_1 + 2$, so $d'_1 \ge d_1 + 3$. Then define $\mathbf{e}$ to be the following hook partition:

$$e_1 = d_1 + 1$$
$$e'_1 = d'_1 - 1$$

We clearly have that $\mathbf{d} < \mathbf{e}$, $s(\mathbf{e}) = 1$, and $\alpha_1(\mathbf{e}) = 1 + e_1 - e'_1 = 3 + d_1 - d'_1 \le 0$, so $\mathbf{e} \in \mathrm{EG}(N)$. Thus $s(\mathbf{e}) = s(\mathbf{d})$ and from the construction of $\mathbf{e}$ it follows that $\mathbf{e}$ satisfies $(**)$.

Schematically:

$\mathbf{d} =$  $\longrightarrow$ $\mathbf{e} =$ 

This completes the proof of Proposition 2.8. $\square$

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