A short proof of the Littlewood-Richardson rule

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1 Introduction

The Littlewood-Richardson rule is a combinatorial rule for computing the multiplicities of the irreducible polynomial characters of GL_n in the product of two irreducible polynomial characters. It can also be stated in terms of Schur functions as follows. (For undefined terminology see [4, Ch. 1].) For partitions $\lambda \supseteq \mu$ we denote by $s_{\lambda/\mu}$ the skew Schur function indexed by λ/μ. For a (column-strict) tableau T (with positive integer entries) we define the *word* of T, $w(T)$, to be the word obtained from T by reading the entries in T from right to left in successive rows, starting with the top row. Let $\langle -,- \rangle$ be the usual scalar product in the vector space of symmetric functions [4, §1.4].

Definition 1.1. Let $w = w_1 \dots w_n$ be a word in the symbols $1, 2, 3, \dots$ and $\theta = (\theta_1, \theta_2, \dots)$ a partition. For $1 \le r \le n$ and $i \ge 1$ denote by $w_{\le r}$ the word $w_1 \dots w_r$ and by $N(i, w_{\le r})$ the number of occurrences of the symbol i in $w_{\le r}$. We say that w is a (θ, i)-*lattice permutation* if for $1 \le r \le n$ the inequality

$$N(i+1, w_{\le r}) + \theta_{i+1} \le N(i, w_{\le r}) + \theta_i$$

holds. We say that w is a θ-*lattice permutation* if it is a (θ, i)-lattice permutation for all $i \ge 1$.

Note that if $\theta = 0$, then a θ-lattice permutation is the same as a lattice permutation. Next we state the Littlewood-Richardson rule:

Theorem 1.2. *Let $\lambda \supseteq \mu$ and $\nu \supseteq \theta$ be partitions. Then $\langle s_{\lambda/\mu}, s_{\nu/\theta} \rangle$ is the number of tableaux T of shape λ/μ and weight $\nu - \theta$ such that $w(T)$ is a θ-lattice permutation.*

Note that although in the classical formulation of the Littlewood-Richardson rule one has $\theta = 0$ (see e.g. [4, §1.9]), Theorem 1.2 is not more general since $\langle s_{\lambda/\mu}, s_{\nu/\theta} \rangle =$

$\langle s_{\lambda/\mu}s_\theta, s_\nu\rangle$ for all partitions $\lambda \supseteq \mu$ and $\nu \supseteq \theta$. A generalization of the Littlewood-Richardson rule to flagged Schur functions can be found in [5].

Using the machinery of Gelfand-Tsetlin schemes Berenstein and Zelevinsky [2] found an involution type proof of a rule which can be decoded in terms of the usual Littlewood-Richardson rule [3]. In §2 we give an involution type proof of the Littlewood-Richardson rule using the traditional machinery of Young tableaux.

2 Proof of Theorem 1.2

For $r \in \mathbf{Z}$ let h_r denote the rth homogeneous symmetric function. (Recall that if $r < 0$, then $h_r = 0$.) For a partition π denote by $\operatorname{sgn}(\pi)$ the sign of π. Let l be the length of ν. For $\pi \in S_l$ let $\pi(\nu)$ be the sequence $\{\nu_{\pi(j)} - \pi(j) + j\}_{j=1}^{l}$. For an integer sequence $\alpha = (\alpha_1, \alpha_2, ...)$ let

$$\operatorname{Tab}(\lambda/\mu, \alpha) = \{P \mid P \text{ is a tableau of shape } \lambda/\mu \text{ and weight } \alpha\}$$

and $h_\alpha = h_{\alpha_1}h_{\alpha_2}\cdots$. (Note that if α contains a negative entry, then $\operatorname{Tab}(\lambda/\mu, \alpha) = \emptyset$ and $h_\alpha = 0$.) Let

$$\begin{aligned} A &= \{(\pi, P) \mid \pi \in S_l \text{ and } P \in \operatorname{Tab}(\lambda/\mu, \pi(\nu) - \theta)\} \text{ and}\\ B &= \{(\pi, P) \in A \mid w(P) \text{ is not a } \theta\text{-lattice permutation}\}. \end{aligned}$$

Proposition 2.1. *There exists an involution $\phi : B \to B$ such that if $\phi(\pi, P) = (\sigma, Q)$, then $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\pi)$.*

Proof. Let $(\pi, P) \in B$ and $w = w(P)$ be the word of P. Let

$$r = \min\{j \geq 1 \mid w_{\leq j} \text{ is not a } \theta\text{-lattice permutation}\}. \tag{1}$$

Then $w_r = i + 1$ for some $i \geq 1$ and $w_{\leq r}$ is not a (θ, i)-lattice permutation, but it is a (θ, j)-lattice permutation for any $j \neq i$. Note also that

$$N(i + 1, w_{\leq r}) + \theta_{i+1} = N(i, w_{\leq r}) + \theta_i + 1. \tag{2}$$

Call an element i (resp. $i+1$) of P *free*, if there is no $i+1$ (resp. i) in its column. Let R (resp. C) denote the row (resp. column) of w_r as an element of P and let $\tilde{R}$ denote the row above R. We claim that all $(i + 1)$'s in R which are in $w_{>r}$ are free. Assume the contrary. Then there is an i in $\tilde{R}$ which is in a column to the left of C. Since the rows of P are weakly increasing and the columns are strictly increasing, it follows that none of the $(i + 1)$'s in R in columns weakly to the right of C are free and that there are no $(i + 1)$'s in $\tilde{R}$ in columns weakly to the left of C. This shows that if w_u is the i

in $\tilde{R}$ above the rightmost $(i+1)$ in R, then all $(i+1)$'s in the word $w_u \dots w_r$ are paired with some of the i's in $w_u \dots w_r$, so $N(i+1, w_u \dots w_r) \leq N(i, w_u \dots w_r)$. Therefore

$$N(i+1, w_{\leq(u-1)}) - N(i, w_{\leq(u-1)}) \geq N(i+1, w_{\leq r}) - N(i, w_{\leq r}) > \theta_i - \theta_{i+1},$$

so $w_{\leq u-1}$ is not a θ-lattice permutation. This contradicts (1) since $u - 1 < r$.

Now we are ready to construct ϕ. Let $\sigma = \pi \circ (i, i+1)$, where $(i, i+1)$ is the permutation that interchanges i and $i+1$. Let Q be obtained from P by replacing all free i's (resp. $(i+1)$'s) in P that are in $w_{>r}$ with $(i+1)$'s (resp. i's) and arranging them in weakly increasing order, independently in each row. Then Q is a column-strict tableau such that

$$\begin{aligned} &r = \min\{j \geq 1 \mid w(Q)_{\leq j} \text{ is not a } \theta\text{-lattice permutation}\} \text{ and} \\ &w(Q)_r = i+1. \end{aligned} \tag{3}$$

The fact that all $(i+1)$'s in R which are in $w_{>r}$ are free implies that

$$\begin{aligned} &N(i, w(Q)_{>r}) = N(i+1, w(P)_{>r}) \text{ and} \\ &N(i+1, w(Q)_{>r}) = N(i, w(P)_{>r}). \end{aligned} \tag{4}$$

Note that for $1 \leq j \leq l$ we have

$$\sigma(\nu)_j = \begin{cases} \pi(\nu)_{i+1} - 1, & \text{if } j = i; \\ \pi(\nu)_i + 1, & \text{if } j = i+1; \\ \pi(\nu)_j, & \text{otherwise.} \end{cases} \tag{5}$$

From (2), (4), and (5) it follows that the weight of Q is $\sigma(\nu) - \theta$, so $(\sigma, Q) \in B$. Define $\phi(\pi, P) = (\sigma, Q)$. Note that if we replace all free i's (resp. $(i+1)$'s) in Q that are in $w(Q)_{>r}$ with $(i+1)$'s (resp. i's) and arrange them in weakly increasing order, independently in each row, then we obtain P. This fact together with (3) shows that $\phi(\sigma, Q) = (\pi, P)$, i.e., ϕ is an involution. We also have that $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\pi)$ which completes the proof of Proposition 2.1. $\square$

Remark 2.2. A similar involution has been used by Bender and Knuth to prove that the skew Schur functions are symmetric [1].

Proof of Theorem 1.2. It is well known that $s_{\lambda/\mu} = \sum_P x^P$, where P runs over all tableaux of shape λ/μ, and

$$s_{\nu/\theta} = \det(h_{\nu_i - \theta_j - i + j})_{1 \leq i,j \leq l} = \sum_{\pi \in S_l} \operatorname{sgn}(\pi) h_{\pi(\nu) - \theta} \,.$$

For a finite set S denote by $|S|$ its cardinality. We have that

$$\begin{aligned}\langle s_{\lambda/\mu}, s_{\nu/\theta}\rangle &= \sum_{\pi\in S_l} \operatorname{sgn}(\pi)\langle s_{\lambda/\mu}, h_{\pi(\nu)-\theta}\rangle = \sum_{\pi\in S_l} \operatorname{sgn}(\pi)|Tab(\lambda/\mu, \pi(\nu)-\theta)| \\ &= \sum_{(\pi,P)\in A} \operatorname{sgn}(\pi) = \sum_{(\pi,P)\in A\setminus B} \operatorname{sgn}(\pi) + \sum_{(\pi,P)\in B} \operatorname{sgn}(\pi)\,.\end{aligned}$$

Note that $(\pi, P) \in A \setminus B$ implies that w is a θ-lattice permutation. In particular, for $i \geq 1$ we have that

$$\pi(\nu)_{i+1} = N(i+1, w) + \theta_{i+1} \leq N(i, w) + \theta_i = \pi(\nu)_i$$

which implies that π is the identity permutation. Therefore $\sum_{(\pi,P)\in A\setminus B} \operatorname{sgn}(\pi)$ is the number of tableaux of shape λ/μ and weight $\nu - \theta$ whose words are θ-lattice permutations. To complete the proof of Theorem 1.2 it remains to show that $\sum_{(\pi,P)\in B} \operatorname{sgn}(\pi) = 0$. This follows immediately from Proposition 2.1. $\square$

Acknowledgment. This paper contains part of the author's Ph. D. thesis written under the supervision of Professor Ira Gessel. The author is grateful to Professor Gessel for his support and encouragement and for pointing out a simplification in the first version of the proof of Theorem 1.2.

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