On the Neggers-Stanley Conjecture and the Eulerian Polynomials

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Abstract

We prove combinatorially that the W-polynomials of naturally labeled graded posets of rank 1 or 2 (an antichain has rank 0) are unimodal, thus providing further supporting evidence for the Neggers-Stanley conjecture. For such posets we also obtain a combinatorial proof that the W-polynomials are symmetric. Combinatorial proofs that the Eulerian polynomials are log-concave and unimodal are given and we construct a simplicial complex Δ with the property that the Hilbert function of the exterior algebra modulo the Stanley-Reisner ideal of Δ is the sequence of Eulerian numbers, thus providing a combinatorial proof of a result of Brenti.

1 Introduction

Let $P = (P, \prec)$ be a poset with n elements and $\lambda : P \to \{1, 2, \ldots, n\}$ a labeling of the vertices of P. A permutation $\omega = \omega_1 \ldots \omega_n \in S_n$ is said to be (P, λ)-*compatible* if $\lambda^{-1}(\omega_i) \prec_P \lambda^{-1}(\omega_j)$ implies $i < j$. For $i \geq 1$ let $d_i(P, \lambda)$ be the number of (P, λ)-compatible permutations with i descents. The W-polynomial of the pair (P, λ) is defined by

$$W(P, \lambda, t) = \sum_{i \geq 0} d_i(P, \lambda) t^{i+1}. \tag{1}$$

Two labelings λ and μ of P are called *equivalent* if $\lambda(x) < \lambda(y) \iff \mu(x) < \mu(y)$ for all $x, y \in P$ such that y covers x. It can be shown that as a function of λ, $W(P, \lambda, t)$ depends only on the equivalence class of λ. If P is a graded poset of rank 0 (i.e., an antichain), then $W(P, \lambda, t) = A_n(t)$, the nth Eulerian polynomial. It is well known that the Eulerian polynomials have only real zeros. There is a conjectured generalization of this fact due to Neggers and Stanley (see [2, 12, 13, 18]):

Conjecture 1.1. *For any labeled poset (P, λ), the polynomial $W(P, \lambda, t)$ has only real zeros.*

A labeling λ of a poset P is called *natural* if $\lambda(u) < \lambda(v)$ for any $u, v \in P$ with $u \prec_P v$. In §2 we prove the following:

Theorem 1.2. *If P is a graded poset with $1 \leq \operatorname{rank} P \leq 2$ and λ a natural labeling of P, then $W(P, \lambda, t)$ is unimodal.*

The proof that we provide for Theorem 1.2 can be considered combinatorial, although we do not explicitly exhibit the necessary injections as this would be rather cumbersome. Since any polynomial with positive coefficients and only real zeros is unimodal, Theorem 1.2 provides further supporting evidence for Conjecture 1.1. It will be interesting to generalize Theorem 1.2 to naturally labeled graded posets of arbitrary rank.

Part of the goal of this paper is to provide direct combinatorial proofs of some well known results on the Eulerian polynomials and a class of W-polynomials. More precisely, we have:

- It is easy to construct a bijection which proves that the Eulerian polynomials are symmetric (see e.g. §6). The following generalization of this fact is known [11, 14]:

  Theorem 1.3. *If P is a graded poset on n elements and λ a natural labeling of P, then $W(P, \lambda, t)$ is symmetric with center of symmetry at $\frac{n - \operatorname{rank} P + 1}{2}$.*

  In §2 we give a proof of Theorem 1.3 in the case $1 \leq \operatorname{rank} P \leq 2$ which, as the unimodality proof mentioned above, can be considered combinatorial.

- From the theory of total positivity (see e.g. [1, 2, 3]) we know that a polynomial with nonnegative coefficients has only real zeros exactly when all minors of the associated Toeplitz matrix are nonnegative. In particular, for $n \geq 1$ the 2×2 minors of the Toeplitz matrix M_n associated with the Eulerian polynomial $A_n(t)$ are nonnegative, hence $A_n(t)$ is log-concave. In §4 we construct an injection ψ which proves the nonnegativity of the 2×2 minors. To do this we first establish a bijection in §3 between permutations and certain labeled lattice paths and then we use a modification of the non-intersecting lattice paths method of Gessel [7] and Gessel-Viennot [9, 10]. It will be very interesting to find injections which show that all minors of M_n are nonnegative; the injection ψ does not permit a straightforward generalization to arbitrary minors.

- It is known that the Eulerian polynomials can be written in the form

  $$A_n(t) = \sum_{l=1}^{\lfloor \frac{n+1}{2} \rfloor} c_{n,l} t^l (1+t)^{n-2l+1}, \qquad (2)$$

  where $c_{n,l}$ is the number of increasing binary trees on n vertices with exactly l leaves and no vertices with left children only. This formula shows that the Eulerian polynomials are unimodal. Gessel [8] has observed that (2) can be proved combinatorially by using a representation of permutations as increasing binary trees (see e.g. [14, pp.23-24]). The proof can also be gleaned from [6]. Stembridge [17, Corollary 2.3] has given another combinatorial proof of this fact by interpreting permutations as certain marked sequences of nonnegative integers [17, Theorem 1.1]. In §5 we construct an injection working directly with permutations which proves the unimodality of the Eulerian polynomials.

- For $1 \leq k \leq n$, the Eulerian number $A(n,k)$ is the coefficient of t^k in $A_n(t)$. In [4, Corollary 4.8] Brenti showed that the sequence $\{A(n,k)\}_{k=1}^n$ of Eulerian numbers is a Hilbert function of a standard graded algebra over a field. Using the interpretation of permutations as labeled lattice paths given in §3 we construct in §6 a simplicial complex Δ whose $(k-1)$-dimensional faces correspond to permutations with k descents. This provides a combinatorial proof of Brenti's result, since it shows that $\{A(n,k)\}_{k=1}^n$ is the Hilbert function of the exterior algebra modulo the Stanley-Reisner ideal of Δ. Edelman and Reiner [5] and Stanley [16] have also found simplicial complexes whose $(k-1)$-dimensional faces correspond to permutations with k descents.

2 Graded posets of small rank

From now on we assume that P is a graded poset and λ a natural labeling of P. Since any two natural labelings of P are equivalent, we will use the term P-compatible instead of (P,λ)-compatible and we will write $W(P,t)$ for $W(P,\lambda,t)$.

Let ρ denote the rank function of P. Fix a labeling λ such that if $a, b \in P$ and $\rho(a) < \rho(b)$, then $\lambda(a) < \lambda(b)$. (Any such labeling is a natural labeling.) Let π be a P-compatible permutation. We can write π uniquely in the form $\pi = \pi_1 \ldots \pi_k$, where $\pi_1, \ldots, \pi_k$ are words such that for $1 \leq i \leq k$ and elements a and b in π_i we have $\rho(\lambda^{-1}(a)) = \rho(\lambda^{-1}(b))$ and for $1 \leq i < k$ and elements a in π_i and b in π_{i+1} we have $\rho(\lambda^{-1}(a)) \neq \rho(\lambda^{-1}(b))$. We will call $\pi_1, \ldots, \pi_k$ the *blocks* of π. For $1 \leq i \leq k$ let $L(\pi_i)$ be the set of elements of π_i and $\rho(\pi_i) = \rho(\lambda^{-1}(a))$, where a is any element of π_i. Let $r = \text{rank } P + 1$ and $\|\pi\| = \{L(\pi_1), \ldots, L(\pi_k)\}$, so $\|\pi\|$ is a partition of $\{1, \ldots, n\}$. For $1 \leq j \leq r$ let ρ_j be the number of π_i's, $1 \leq i \leq k$, with $\rho(\pi_i) = j-1$. Denote by $\hat{\pi}$ the permutation of the multiset $R_\pi = \{1^{\rho_1}, \ldots, r^{\rho_r}\}$ obtained from π by replacing each block π_i with the number $\rho(\pi_i)+1$. For a permutation σ of a multiset denote by $d(\sigma)$ the number of (strict) descents of σ. If π_i and π_j are two blocks of π with $\rho(\pi_i) < \rho(\pi_j)$, then for any a in π_i and b in π_j we have $\lambda(a) < \lambda(b)$. This shows that

$$d(\pi) = d(\hat{\pi}) + \sum_{i=1}^{k} d(\pi_i). \tag{3}$$

Note also that no two adjacent elements of $\hat{\pi}$ are the same. Call a permutation that has this property *allowable*. If $|\rho(\pi_{i+1}) - \rho(\pi_i)| \geq 2$ for some i, then no element of π_i

is comparable with an element of π_{i+1}, so the permutation $\pi_1 \dots \pi_{i-1}\pi_{i+1}\pi_i\pi_{i+2}\dots\pi_k$ is P-compatible. This suggests the following definition:

Definition 2.1. Two allowable permutations σ and τ of a multiset M are called *similar*, if there exists a sequence $\sigma = \omega^{(1)}, \omega^{(2)}, \dots, \omega^{(r)} = \tau$ of permutations of M (not necessarily allowable), such that for $1 \le i \le r-1$, $\omega^{(i+1)}$ is obtained from $\omega^{(i)}$ by interchanging two adjacent elements a and b with $|b-a| \ge 2$.

Note that this relation is an equivalence relation. If σ is a permutation of a multiset, let $A_\sigma(t) = \sum_\tau t^{d(\tau)+1}$, where the sum is taken over all permutations τ which are similar to σ. Two P-compatible permutations π and σ are called *P-equivalent*, if $\|\pi\| = \|\sigma\|$ and $\hat{\pi}$ and $\hat{\sigma}$ are similar. (This is also an equivalence relation.) Let $\pi^{(1)}, \dots, \pi^{(m)}$ be representatives of the P-equivalence classes of the set of P-compatible permutations. For $1 \le i \le m$ let $\pi_1^{(i)}, \dots, \pi_{k_i}^{(i)}$ be the blocks of $\pi^{(i)}$. For $1 \le i \le m$ and $1 \le j \le k_i$ denote by n_{ij} the number of elements of $\pi_j^{(i)}$. By (3) we deduce the following:

Proposition 2.2. *With the above notation and hypothesis we have the decomposition*

$$W(P,t) = \sum_{i=1}^{m} A_{\widehat{\pi^{(i)}}}(t) \frac{A_{n_{i1}}(t)}{t} \dots \frac{A_{n_{ik_i}}(t)}{t}. \tag{4}$$

Proof of Theorem 1.2 and the case $1 \le \operatorname{rank} P \le 2$ *of Theorem 1.3.*

First consider the case $\operatorname{rank} P = 1$. Let $\pi = \pi_1 \dots \pi_k$ be a P-compatible permutation with blocks $\pi_1, \dots, \pi_k$. We must have $\rho(\pi_1) = 0$ and $\rho(\pi_k) = 1$, so $2 \mid k$ and

$$\rho(\pi_i) = \begin{cases} 0, & \text{if } 2 \nmid i; \\ 1, & \text{if } 2 \mid i. \end{cases}$$

This shows that $A_{\hat{\pi}}(t) = t^{\frac{k}{2}}$, so $A_{\hat{\pi}}(t)$ is symmetric and unimodal with center of symmetry at $\frac{k}{2}$. For $1 \le j \le k$ let n_j be the number of elements of π_j. Since each Eulerian polynomial $A_n(t)$ is symmetric and unimodal with center of symmetry at $\frac{n+1}{2}$, it follows that $A_{\hat{\pi}}(t)\frac{A_{n_1}(t)}{t} \dots \frac{A_{n_k}(t)}{t}$ is symmetric and unimodal with center of symmetry at $\frac{k}{2} + \sum_{j=1}^{k} \frac{n_j - 1}{2} = \frac{\sum_{j=1}^{k} n_j}{2} = \frac{n}{2}$. By Proposition 2.2 it follows that $W(P,t)$ is symmetric and unimodal with center of symmetry at $\frac{n}{2} = \frac{n - \operatorname{rank} P + 1}{2}$.

Now assume that $\operatorname{rank} P = 2$ and π is a P-compatible permutation. Then $\hat{\pi}$ has the form $\hat{\pi} = 1\,2\,\sigma_1\,2\,\sigma_2\,2 \dots 2\,\sigma_r\,2\,3$, where for $1 \le i \le r$, σ_i is an allowable permutation of a multiset $\{1^{a_i}, 3^{b_i}\}$ with $|a_i - b_i| \le 1$. For $1 \le i \le r$ let $\tilde{\sigma}_i = 2\,\sigma_i\,2$ and c_i be the number of elements of σ_i. If $a_i \ne b_i$, then $\tilde{\sigma}_i$ is the only permutation in its equivalence class. If $a_i = b_i$, then the equivalence class of $\tilde{\sigma}_i$ contains $\tilde{\sigma}_i$ and the permutation obtained from $\tilde{\sigma}_i$

by replacing each occurrence of 1 with a 3 and each occurrence of 3 with a 1. This shows that for $1 \le i \le r$

$$A_{\tilde{\sigma}_i}(t) = \begin{cases} t^{\frac{c_i+3}{2}}, & \text{if } a_i \neq b_i; \\ t^{\frac{c_i+2}{2}}(1+t), & \text{if } a_i = b_i. \end{cases}$$

In particular, each $A_{\tilde{\sigma}_i}(t)$ is symmetric and unimodal with center of symmetry at $\frac{c_i+3}{2}$. Let $\pi_1, \ldots, \pi_k$ be the blocks of π. We have that $A_{\hat{\pi}}(t) = t\prod_{i=1}^{r} \frac{A_{\tilde{\sigma}_i}(t)}{t}$, so $A_{\hat{\pi}}(t)$ is symmetric and unimodal with center of symmetry at $1 + \sum_{i=1}^{r} \frac{c_i+1}{2} = \frac{k-1}{2}$. If for $1 \le i \le k$, π_i has n_i elements, then $A_{\hat{\pi}}(t)\frac{A_{n_1}(t)}{t} \ldots \frac{A_{n_k}(t)}{t}$ is symmetric and unimodal with center of symmetry at

$$\frac{k-1}{2} + \sum_{i=1}^{k} \frac{n_i - 1}{2} = \frac{\left(\sum_{i=1}^{k} n_i\right) - 1}{2} = \frac{n-1}{2}.$$

By Proposition 2.2 it follows that $W(P,t)$ is symmetric and unimodal with center of symmetry at $\frac{n-1}{2} = \frac{n-\text{rank } P+1}{2}$. □

3 The labeled paths interpretation

Let $n \ge 1$ and $0 \le k \le n-1$. We will denote by $\mathcal{A}(n, k+1)$ the set of permutations in S_n with k descents and by $\mathcal{P}(n,k)$ the set of labeled northeastern lattice paths P with $n-1$ steps, of which exactly k are vertical, such that a horizontal step $i-1$ units above the initial point of P is labeled with an integer between 1 and i, and similarly a vertical step $i-1$ units to the right of the initial point of P is labeled with an integer between 1 and i. (We do not distinguish between paths that can be obtained from each other by a translation.)

Proposition 3.1. *For any n and k such that $n \ge 1$ and $0 \le k \le n-1$, there exists a bijection $\Phi_{n,k}$ between the sets $\mathcal{A}(n, k+1)$ and $\mathcal{P}(n,k)$.*

Proof. We argue by induction on n. If $n = 1$, then we must have $k = 0$. We have that $|\mathcal{A}(1,1)| = |\mathcal{P}(1,0)| = 1$ which completes the proof in the case $n = 1$. Now let $n > 1$ and suppose that for $0 \le k \le n-2$ we have already constructed the bijections $\Phi_{n-1,k}$. Fix $0 \le k \le n-1$ and let $\pi \in \mathcal{A}(n, k+1)$. Let $\pi' \in S_{n-1}$ be the permutation obtained from π by removing the letter n. Then $\pi' \in \mathcal{A}(n-1, k+1)$ (this is only possible when $k \le n-2$) or $\pi' \in \mathcal{A}(n-1,k)$. Let P' be $\Phi_{n-1,k+1}(\pi')$ in the first case and $\Phi_{n-1,k}(\pi')$ in the second. We will define $P \in \mathcal{P}(n,k)$ as follows. If $\pi' \in \mathcal{A}(n-1,k+1)$, then there are exactly $k+1$ positions $p_1, \ldots, p_{k+1}$ (ordered from left to right) to insert n in π' and obtain a permutation in $\mathcal{A}(n, k+1)$. If n has to be inserted in position p_i to obtain π, then we obtain P by adding a horizontal step at the northeastern endpoint of P' and labeling it

i. If $\pi' \in \mathcal{A}(n-1,k)$, then there are exactly $n-k$ positions $q_1,\ldots,q_{n-k}$ (ordered again from left to right) to insert n in π' and obtain a permutation in $\mathcal{A}(n,k+1)$. If n has to be inserted in position q_i to obtain π, then we obtain P by adding a vertical step at the northeastern endpoint of P' and labeling it i. Then we let $\Phi_{n,k}(\pi) = P$. (See Fig. 1 for an example.)

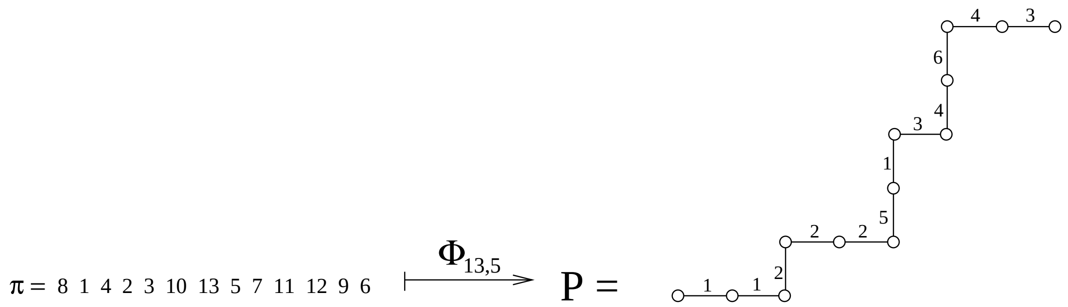

Figure 1.

To see that $\Phi_{n,k}$ is a bijection, let $P \in \mathcal{P}(n,k)$ and P' be the path obtained from P by removing the last step (the steps are ordered from southwest to northeast). By the inductive hypothesis, P' corresponds to a permutation $\pi' \in S_{n-1}$ via $\Phi_{n-1,k}$ or $\Phi_{n-1,k-1}$. First assume that the last step of P is horizontal. Then $\pi' \in \mathcal{A}(n-1,k+1)$ and there are exactly $k+1$ positions $p_1,\ldots,p_{k+1}$ to insert n in π' and obtain a permutation in $\mathcal{A}(n,k+1)$. If the last step of P is labeled i, then let π be the permutation obtained from π' by inserting n in position p_i. Then $\Phi_{n,k}(\pi) = P$. Now assume that the last step of P is vertical. Then $\pi' \in \mathcal{A}(n-1,k)$ and there are exactly $n-k$ positions $q_1,\ldots,q_{n-k}$ to insert n in π' and obtain a permutation in $\mathcal{A}(n,k+1)$. If the last step is labeled i, then let π be the permutation obtained from π' by inserting n in position q_i. Then $\Phi_{n,k}(\pi) = P$. $\square$

4 A combinatorial proof of the log-concavity of the Eulerian polynomials

Let P be a lattice path. We will say that a horizontal step H of P is at level i in P if H is $i-1$ units above the initial point of P. We will denote the level of H in P by $r_P(H)$. Note that the range of labels that can be assigned to H as a step in P is $\{1,\ldots,r_P(H)\}$.

Theorem 4.1. *For $n \geq 1$ the Eulerian polynomial $A_n(t)$ is log-concave.*

Proof. To prove the theorem combinatorially we will construct an injection

$$\psi : \mathcal{P}(n,k-1) \times \mathcal{P}(n,k+1) \hookrightarrow \mathcal{P}(n,k) \times \mathcal{P}(n,k)$$

for $n \geq 1$ and $1 \leq k \leq n-2$. Let $(P,Q) \in \mathcal{P}(n,k-1) \times \mathcal{P}(n,k+1)$. Place the initial points of P and Q at $(0,0)$ and $(1,-1)$ respectively. Then the endpoints of P and Q

are $(n-k+1,k-1)$ and $(n-k,k)$ respectively, so P and Q intersect. Let A be the last intersection point (we order the intersection points from southwest to northeast) and decompose $P = P_1 \cup P_2$ and $Q = Q_1 \cup Q_2$, where P_1 is a path from $(0,0)$ to A, P_2 is a path from A to $(n-k+1,k-1)$, Q_1 is a path from $(1,-1)$ to A, and Q_2 is a path from A to $(n-k,k)$. Let $P' = P_1 \cup R$ and $Q' = Q_1 \cup S$, where R and S are labeled paths from A to $(n-k,k)$ and from A to $(n-k+1,k-1)$ respectively, defined as follows. As unlabeled paths R coincides with Q_2 and S coincides with P_2. Let $H_1,\ldots,H_{s+1}$ be the horizontal steps of P_2 and $G_1,\ldots,G_s$ the horizontal steps of Q_2. Then

$$\begin{aligned} r_{Q'}(H_i) &= r_P(H_i) + 1 \text{ and} \\ r_{P'}(G_i) &= r_Q(G_i) - 1 \end{aligned}$$

for $i = 1,2,\ldots,s$. Let l_i, $i = 1,2,\ldots,s+1$, be the label of H_i as a horizontal step in P and let m_i, $i = 1,2,\ldots,s$, be the label of G_i as a horizontal step in Q. Then we label H_{s+1} as a horizontal step in Q' by l_{s+1}. For $1 \le i \le s$ we label H_i and G_i as horizontal steps in Q' and P' by l'_i and m'_i respectively, where

$$(l'_i, m'_i) = \begin{cases} (l_i, m_i), & \text{if } m_i < r_Q(G_i); \\ (r_{Q'}(H_i), l_i), & \text{if } m_i = r_Q(G_i). \end{cases}$$

(We can do this since $r_P(H_i) \le r_Q(G_i) - 2$.) We label the vertical steps of R and S in the same way by viewing them as horizontal steps of the paths which are symmetric to P and Q with respect to the line of slope 1 passing through $(1,0)$. Then we define $\psi(P,Q) = (P',Q')$. (See Fig. 2 for an example.)

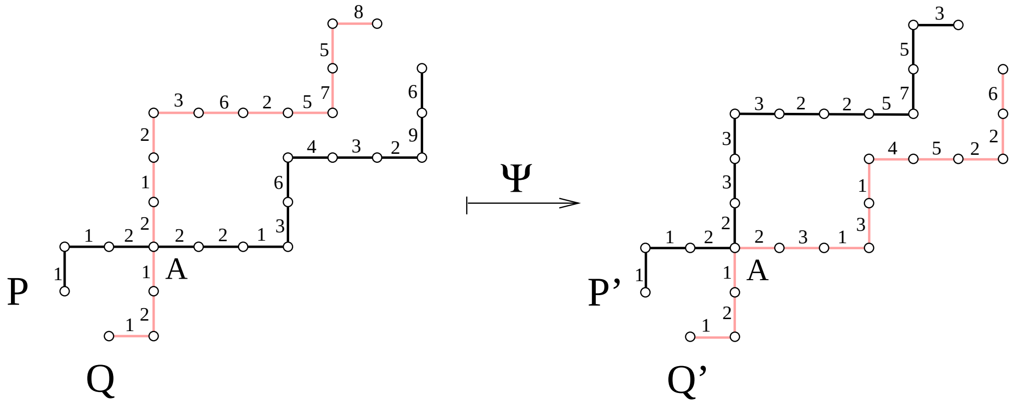

Figure 2.

We will show that ψ is a bijection between $\mathcal{P}(n,k-1) \times \mathcal{P}(n,k+1)$ and the subset U of pairs $(P',Q') \in \mathcal{P}(n,k) \times \mathcal{P}(n,k)$ satisfying the following conditions. Place the initial points of P' and Q' at $(0,0)$ and $(1,-1)$ respectively (so the endpoints are $(n-k,k)$ and $(n-k+1,k-1)$ respectively). Then:

1. P' and Q' intersect;

2. Let A be the last intersection point of P' and Q' and decompose $P' = P_1' \cup P_2'$ and $Q' = Q_1' \cup Q_2'$, where P_1' is a path from $(0,0)$ to A, P_2' is a path from A to $(n-k,k)$, Q_1' is a path from $(1,-1)$ to A, and Q_2' is a path from A to $(n-k+1,k-1)$. Let $H_1,\dots,H_{s+1}$ be the horizontal steps of Q_2' labeled $l_1',\dots,l_{s+1}'$ respectively. Let $G_1,\dots,G_s$ be the horizontal steps of Q_2' labeled $m_1',\dots,m_s'$ respectively. Then $l_{s+1}' < r_{Q'}(H_{s+1})$ and for $1 \le i \le s$, $l_i' < r_{Q'}(H_i)$ or $l_i' = r_{Q'}(H_i)$ and $m_i' < r_{Q'}(H_i)$.

3. Condition (2) holds for the paths which are symmetric to P' and Q' with respect to the line of slope 1 passing through $(1,0)$.

By the construction of ψ it follows that $\psi(\mathcal{P}(n,k-1)\times\mathcal{P}(n,k+1)) \subseteq U$. Define a map $\varphi : U \to \mathcal{P}(n,k-1)\times\mathcal{P}(n,k+1)$ as follows. Let $(P',Q') \in U$. In the notation we used to describe U let $P = P_1' \cup R'$, where R' is a labeled path from A to $(n-k+1,k-1)$ such that $R' = Q_2'$ as unlabeled paths. Let $Q = Q_1' \cup S'$, where S' is a labeled path from A to $(n-k,k)$ such that $S' = P_2'$ as unlabeled paths. Then $H_1,\dots,H_{s+1}$ are the horizontal steps of R' and $G_1,\dots,G_s$ are the horizontal steps of S'. We label H_{s+1} as a horizontal step in P by l_{s+1}'. For $1 \le i \le s$ we label H_i and G_i as horizontal steps in P and Q by l_i and m_i respectively, where

$$(l_i,m_i) = \begin{cases} (l_i',m_i'), & \text{if } l_i' < r_{Q'}(H_i); \\ (m_i', r_{P'}(G_i)+1), & \text{if } l_i' = r_{Q'}(H_i). \end{cases}$$

We label the vertical steps of R' and S' in the same way by viewing them as horizontal steps of the paths which are symmetric to P' and Q' with respect to the line of slope 1 passing through $(1,0)$. By conditions (1), (2), and (3) it follows that $P \in \mathcal{P}(n,k-1)$ and $Q \in \mathcal{P}(n,k+1)$. We define $\varphi(P',Q') = (P,Q)$. Then $\varphi\psi(P,Q) = (P,Q)$ and $\psi\varphi(P',Q') = (P',Q')$ for any $(P,Q) \in \mathcal{P}(n,k-1)\times\mathcal{P}(n,k+1)$ and $(P',Q') \in U$, so ψ is injective. □

Remark 4.2. Unfortunately, the injection ψ in the proof of Theorem 4.1 cannot be used to prove that the W-polynomials are log-concave because it does not preserve P-compatibility. Consider for example the graded poset P with elements $v_1,\dots,v_5$ and relations $v_i < v_j$ for $1 \le i \le 3$, $4 \le j \le 5$. For $1 \le i \le 5$ label the vertex v_i with the number i (this is a natural labeling). The permutations $\pi = 1\,2\,3\,5\,4$ and $\sigma = 3\,2\,1\,5\,4$ are P-compatible with $d(\pi) = 1$ and $d(\sigma) = 3$. Let Q and R be the lattice paths corresponding to π and σ respectively. We have that $\psi(Q,R) = (Q',R')$, where Q' and R' correspond to the permutations $\pi' = 1\,5\,3\,4\,2$ and $\sigma' = 5\,2\,4\,1\,3$ respectively which are not P-compatible.

5 A combinatorial proof of the unimodality of the Eulerian polynomials

It is well known that the Eulerian polynomials $A_n(t)$ are symmetric with center of symmetry at $\frac{n+1}{2}$. This can be proved by exhibiting a bijection $\phi : S_n \to S_n$ which sends permutations with i descents to permutations with $n-1-i$ descents for $0 \le i \le n-1$; e.g. for $\pi = a_1 \dots a_n \in S_n$ we can define $\phi(\pi) = b_1 \dots b_n$, where $b_j = n+1-a_j$ for $1 \le j \le n$. So to prove that the Eulerian polynomials are unimodal, it is enough to show that $A(n,k) \le A(n,k+1)$ for $1 \le k \le \frac{n-1}{2}$. Fix one such k. We will construct an injection

$$\psi = \psi_k : \mathcal{A}(n,k) \hookrightarrow \mathcal{A}(n,k+1).$$

Let $\sigma \in \mathcal{A}(n,k)$. Denote by $a(\sigma)$ the number of ascents of σ and recall that we denote by $d(\sigma)$ the number of descents of σ. We have that $d(\sigma) + a(\sigma) = n-1$ and since $d(\sigma) = k-1 \le \frac{n-3}{2}$, it follows that $a(\sigma) = n-1-d(\sigma) \ge \frac{n+1}{2}$, so $a(\sigma)-d(\sigma) \ge \frac{n+1}{2} - \frac{n-3}{2} = 2$. If $\sigma = b_1 \dots b_n$, then for $1 \le i \le n$ define

$$\begin{aligned} d_i(\sigma) &= |\{j : 1 \le j \le n-1 \text{ and } b_{j+1} < b_j \le i\}| \text{ and} \\ a_i(\sigma) &= |\{j : 1 \le j \le n-1 \text{ and } b_j < b_{j+1} \le i\}|. \end{aligned}$$

In other words, $d_i(\sigma)$ (resp. $a_i(\sigma)$) is the number of descents (resp. ascents) of σ that only involve numbers from $\{1, 2, \dots, i\}$. We have that $d_n(\sigma) = d(\sigma)$, $a_n(\sigma) = a(\sigma)$, and $d_1(\sigma) = a_1(\sigma) = 0$, so $a_n(\sigma) - d_n(\sigma) \ge 2$ and $a_1(\sigma) - d_1(\sigma) = 0$. In addition, for $1 \le i \le n-1$ we have $a_i(\sigma) \le a_{i+1}(\sigma) \le a_i(\sigma) + 1$ and $d_i(\sigma) \le d_{i+1}(\sigma) \le d_i(\sigma) + 1$, so $|(a_{i+1}(\sigma) - d_{i+1}(\sigma)) - (a_i(\sigma) - d_i(\sigma))| \le 1$. This implies that there is a number j, $2 \le j \le n-1$, such that $a_j(\sigma) - d_j(\sigma) = 1$. Let $m(\sigma)$ be the largest such j and τ the word obtained from σ by replacing each r, $1 \le r \le m(\sigma)$, with $m(\sigma) + 1 - r$. Then $\tau \in S_n$ and we have:

$$\begin{aligned} a_{m(\sigma)}(\tau) &= d_{m(\sigma)}(\sigma), \\ d_{m(\sigma)}(\tau) &= a_{m(\sigma)}(\sigma), \\ a(\tau) - a_{m(\sigma)}(\tau) &= a(\sigma) - a_{m(\sigma)}(\sigma), \text{ and} \\ d(\tau) - d_{m(\sigma)}(\tau) &= d(\sigma) - d_{m(\sigma)}(\sigma). \end{aligned} \tag{5}$$

Since $a_{m(\sigma)}(\sigma) - d_{m(\sigma)}(\sigma) = 1$, it follows that $a(\tau) - d(\tau) = (a(\sigma) - d(\sigma)) - 2$. This implies that $d(\tau) = d(\sigma) + 1$, hence $\tau \in \mathcal{A}(n, k+1)$. For example, if $\sigma = 258169473 \in \mathcal{A}(9,4)$, then $m(\sigma) = 5$ and $\tau = 418569273 \in \mathcal{A}(9,5)$. We define $\psi(\sigma) = \tau$.

It remains to show that ψ is injective. Let

$$U = \{\tau \in \mathcal{A}(n,k+1) : \text{there exists } 2 \le j \le n-1 \text{ such that } a_j(\tau) - d_j(\tau) = -1\}.$$

By construction $\psi(\mathcal{A}(n,k)) \subseteq U$. We will show that ψ is a bijection between $\mathcal{A}(n,k)$ and U. For this purpose we will construct a map $\varphi : U \to \mathcal{A}(n,k)$ as follows. Let $\tau \in U$ and define $n(\tau)$ to be the largest integer such that $a_{n(\tau)}(\tau) - d_{n(\tau)}(\tau) = -1$. Let σ be the word obtained from τ by replacing each r, $1 \le r \le n(\tau)$, with $n(\tau) + 1 - r$. Then $\sigma \in S_n$ and the equalities (5) hold with $m(\sigma)$ replaced by $n(\tau)$. Since $a_{n(\tau)}(\tau) - d_{n(\tau)}(\tau) = -1$, it follows that $d(\sigma) = d(\tau) - 1$, so $\sigma \in \mathcal{A}(n,k)$. Then we let $\varphi(\tau) = \sigma$. One easily checks that for $\sigma \in \mathcal{A}(n,k)$ and $\tau \in U$ we have $\varphi\psi(\sigma) = \sigma$ and $\psi\varphi(\tau) = \tau$, so ψ is indeed a bijection between $\mathcal{A}(n,k)$ and U. This completes the proof that the Eulerian polynomials are unimodal.

6 An "Eulerian" simplicial complex

Fix $n \geq 1$. We will construct a simplicial complex Δ whose $(k-1)$-dimensional faces, $1 \leq k \leq n$, correspond to permutations in S_n with k descents. We will identify the elements of $\mathcal{A}(n, k+1)$ with the elements of $\mathcal{P}(n,k)$ via the bijection Φ from §3. We say that a horizontal step in a lattice path Q is on level i, if it is $i-1$ units above the initial point of Q. Let $h(Q)$ denote the number of horizontal steps in Q. If $Q \in \mathcal{P}(n,k)$, then for $1 \leq i \leq k$ we denote by $h_i(Q)$ the number of horizontal steps in Q whose level is at most i. Let $V = \mathcal{P}(n,1)$ be the vertex set of Δ. Let $P \in \mathcal{P}(n,k)$. We will associate to P a k-element subset $\{P_1, \ldots, P_k\}$ of V. For a labeled path P, denote by $\bar{P}$ the path obtained from P by deleting the labels. For $1 \leq i \leq k$ let $\bar{P}_i$ be the unlabeled lattice path with one vertical step, $h_1(\bar{P}_i) = h_i(P) + i - 1$, and $h(\bar{P}_i) = h(P) + k - 1$. Place the initial point of P at $(0,0)$ and for $1 \leq i \leq k$ place the initial point of $\bar{P}_i$ at $(-i+1, i-1)$. (See Fig. 3.)

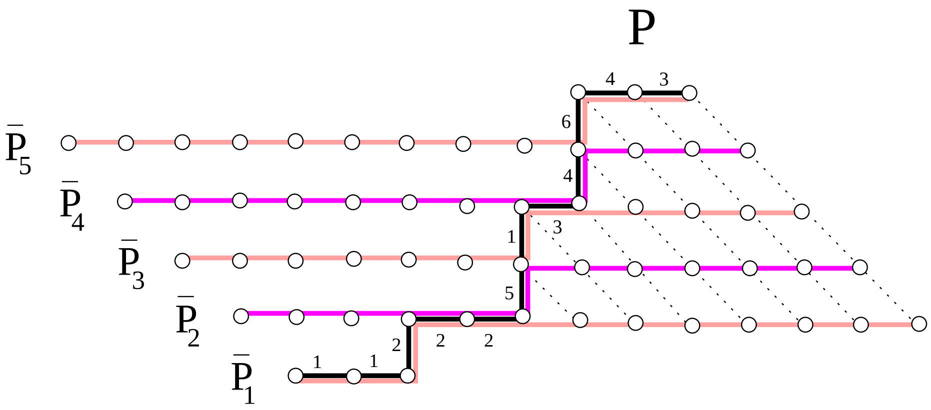

Figure 3.

Note that the vertical step of $\bar{P}_i$, $1 \leq i \leq k$, coincides with the ith vertical step of P and that the range of labels we can assign to the vertical step of $\bar{P}_i$ is not less than the range of labels we can assign to the ith vertical step of P. Label the vertical step of $\bar{P}_i$, $1 \leq i \leq k$, by the label of the ith vertical step of P. Label the level 1 horizontal steps in $\bar{P}_1, \ldots, \bar{P}_k$ by 1. Label the level 2 horizontal steps in $\bar{P}_1$ which are also steps in P by their labels as steps in P. To label the level 2 horizontal steps in $\bar{P}_2, \ldots, \bar{P}_k$, first draw the northwest strips bounded by $\bar{P}_1$ and P as shown on Fig. 3. The northwest border of each such strip is either a vertical or a horizontal step. If it is a vertical step, label all horizontal steps in the strip 1. Now suppose the northwest border of a strip is a horizontal step whose level and label in P are i and j respectively. Then $j \leq i$ and the strip has exactly $i-1$ horizontal steps in it. Order these steps from southeast to northwest and label the first $i-j$ horizontal steps 1 and the remaining $j-1$ horizontal steps 2. (See Fig. 4.)

In this way we obtain paths $P_1, \ldots,\ P_k$ from V. Note that they satisfy the following conditions:

1. $\bar{P}_i \neq \bar{P}_j$ for $1 \leq i \neq j \leq k$.

2. Suppose $P_1, \ldots, P_k$ are ordered such that for $1 \le i \le k-1$, P_i has fewer level 1 horizontal steps than P_{i+1}. (This can be done because of (1).) Then if we draw $P_1, \ldots, P_k$ as in Fig. 4 (so that the initial point of P_{i+1} is one unit up and to the left of the initial point of P_i for $1 \le i \le k-1$), the labels of their level 2 horizontal steps weakly increase in each northwest strip bounded by P_1 and P. When a northwest strip ends with a vertical step, then all horizontal steps in it are labeled 1.

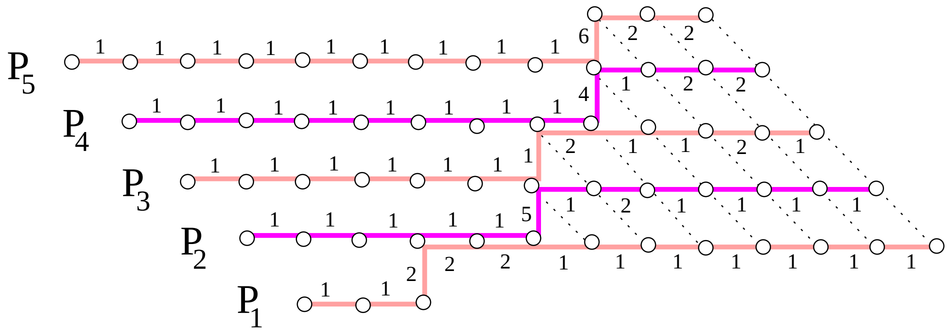

Figure 4.

On the other hand, a set of k paths from V satisfying conditions (1) and (2) above determines a path from $\mathcal{P}(n,k)$. Also, every subset of a set of paths from V satisfying (1) and (2), satisfies (1) and (2) itself. We define Δ to be the set of all subsets of V satisfying (1) and (2). The above discussion shows that Δ is a simplicial complex with the desired properties.

As we mentioned in the Introduction, the existence of Δ proves combinatorially that the sequence $\{A(n,k)\}_{k=1}^n$ of Eulerian numbers is a Hilbert function of a standard graded algebra over a field.

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