Sufficiency of Lakshmibai-Sandhya singularity conditions for Schubert varieties

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Abstract. We establish one direction of a conjecture by Lakshmibai and Sandhya [7] which describes combinatorially the singular locus of a Schubert variety. We prove that the conjectured singular locus is contained in the singular locus.

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1 Introduction

Let n be a positive integer and S_n the group of permutations of the set $\{1, \ldots, n\}$. We use the one-line notation to write the elements of S_n; namely, for $w \in S_n$ we write $w = w_1 w_2 \ldots w_n$, where $w_i = w(i)$ for $1 \leq i \leq n$. We denote the cardinality of a finite set A by $|A|$. Let $\mathrm{Fl}(n)$ be the complete flag variety consisting of flags $\mathbf{E}_\bullet = (E_1 \subset E_2 \subset \cdots \subset E_n = \mathbb{C}^n)$ of nested vector spaces in $\mathbb{C}^n$. For $1 \leq i \leq n$, let e_i be the ith standard basis vector of $\mathbb{C}^n$ and $F_i = \langle e_1, \ldots, e_i \rangle$ the subspace of $\mathbb{C}^n$ spanned by $e_1, \ldots, e_i$. The Schubert variety X_w is the subvariety of $\mathrm{Fl}(n)$ consisting of the flags $\mathbf{E}_\bullet$ such that $\dim(E_p \cap F_q) \geq |\{i \leq p \mid w_i \leq q\}|$ for $1 \leq p, q \leq n$. Equivalently, if B is the Borel subgroup of $GL_n(\mathbb{C})$ consisting of the upper triangular matrices, then $X_w = \overline{BwB/B}$. The Bruhat order $\prec$ on

S_n can be defined as follows:

$$v \preceq w \quad \text{if} \quad |\{i \leq p \mid v_i \leq q\}| \geq |\{i \leq p \mid w_i \leq q\}| \text{ for } 1 \leq p, q \leq n\,.$$

Therefore $v \prec w$ if and only if $X_v \subset X_w$. The Bruhat order makes S_n into a graded poset. The length $l(w)$ of a permutation $w \in S_n$ is the rank of w in the Bruhat order on S_n. Equivalently, $l(w)$ is the number of inversions of w, i.e.,

$$l(w) = |\{(i,j) \mid 1 \leq i < j \leq n \textit{ and } w_i > w_j\}|.$$

We have that $\dim X_w = l(w)$. We associate to $v \in S_n$ the coordinate flag

$$e_v = (\langle e_{v_1}\rangle \subset \langle e_{v_1}, e_{v_2}\rangle \subset \cdots \subset \langle e_{v_1}, \ldots, e_{v_n}\rangle = \mathbb{C}^n)\,.$$

Then $e_v \in X_w$ if and only if $v \preceq w$. For an introduction to the theory of Schubert varieties see e.g. [2].

Smooth Schubert varieties are characterized combinatorially as follows:

Theorem 1.1. (Lakshmibai and Sandhya [7]) *The Schubert variety X_w is smooth if and only if w does not contain a subsequence $w_{i_1}w_{i_2}w_{i_3}w_{i_4}$ of 4 elements with the same relative order as 4231 or 3412.*

Theorem 1.2. (Gasharov [3]) *Let $w \in S_n$. The Schubert variety X_w is smooth if and only if the Poincaré polynomial $p_w(t) = \sum_{v \preceq w} t^{l(v)}$ factors into polynomials of the form $1 + t + t^2 + \cdots + t^r$.*

A criterion for smoothness of Schubert varieties in terms of the nil Hecke ring was given by Kumar [6].

Let $\operatorname{Sing} X_w$ denote the singular locus of X_w. The Borel group B acts on X_w and $\operatorname{Sing} X_w$ is invariant under this action, so $\operatorname{Sing} X_w$ is a union of Schubert varieties X_λ for some $\lambda \prec w$. We have that $\operatorname{Sing} X_{4231} = X_{2143}$ and $\operatorname{Sing} X_{3412} = X_{1324}$ [7, Remark 3.1]. Lakshmibai and Sandhya conjectured a combinatorial description of $\operatorname{Sing} X_w$ in [7]:

Conjecture 1.3. *If $w \in S_n$, then*

$$\operatorname{Sing} X_w = \cup_\lambda X_\lambda\,,$$

where λ runs over all maximal elements (in the Bruhat order) of the set Z consisting of all $\tau' \prec w$ satisfying (1) or (2) below:

(1) There exists a subsequence $w_{i_1}w_{i_2}w_{i_3}w_{i_4}$ of 4 elements in w with the same relative order as 4231. Let $\tau \in S_n$ be the permutation obtained from w by replacing $w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4}$ as elements in w by $w_{i_2}, w_{i_4}, w_{i_1}, w_{i_3}$ respectively. There exists a $w' \in S_n$ containing a subsequence $w'_{j_1}w'_{j_2}w'_{j_3}w'_{j_4}$ such that $w'_{j_1} = w_{i_1}$, $w'_{j_2} = w_{i_2}$, $w'_{j_3} = w_{i_3}$, $w'_{j_4} = w_{i_4}$, τ' is obtained from w' by replacing $w'_{j_1}, w'_{j_2}, w'_{j_3}, w'_{j_4}$ as elements in w' by $w'_{j_2}, w'_{j_4}, w'_{j_1}, w'_{j_3}$ respectively, and $\tau \prec \tau' \prec w' \prec w$.

(2) There exists a subsequence $w_{i_1}w_{i_2}w_{i_3}w_{i_4}$ of 4 elements in w with the same relative order as 3412. Let $\tau \in S_n$ be the permutation obtained from w by replacing $w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4}$ as elements in w by $w_{i_3}, w_{i_1}, w_{i_4}, w_{i_2}$ respectively. There exists a $w' \in S_n$ containing a subsequence $w'_{j_1}w'_{j_2}w'_{j_3}w'_{j_4}$ such that $w'_{j_1} = w_{i_1}$, $w'_{j_2} = w_{i_2}$, $w'_{j_3} = w_{i_3}$, $w'_{j_4} = w_{i_4}$, τ' is obtained from w' by replacing $w'_{j_1}, w'_{j_2}, w'_{j_3}, w'_{j_4}$ as elements in w' by $w'_{j_3}, w'_{j_1}, w'_{j_4}, w'_{j_2}$ respectively, and $\tau \prec \tau' \prec w' \prec w$.

Gonciulea and Lakshmibai showed that Conjecture 1.3 is true for a class of Schubert varieties related to ladder determinantal varieties [4, 5].

A permutation $\pi = \pi_1 \dots \pi_n$ which does not contain a subsequence $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_4}$ with the same relative order as 2143 is called vexillary. The Kazhdan-Lusztig polynomials $P_{\pi,w}(q)$, $\pi \preceq w \in S_n$, measure the singularities of Schubert varieties. In [9] (see also [10]) Lascoux computed the polynomials $P_{\pi,w}(q)$ when π is a vexillary permutation. Other classes of Kazhdan-Lusztig polynomials are treated e.g. in [11, 13].

Next we give an example which illustrates the above conjecture.

Example 1.4. Let $w = 5\,3\,8\,2\,6\,4\,7\,1 \in S_8$. Then the irreducible components of $\operatorname{Sing} X_w$ are the Schubert varieties $X_{\pi^{(i)}}$, $i = 1, 2, 3, 4$, where $\pi^{(1)} = 3\,2\,5\,4\,8\,6\,7\,1$, $\pi^{(2)} = 3\,2\,8\,1\,6\,5\,7\,4$, $\pi^{(3)} = 5\,3\,2\,1\,8\,6\,7\,4$, and $\pi^{(4)} = 5\,3\,6\,2\,4\,1\,8\,7$. We have that $\pi^{(1)}$ satisfies condition (2) of Conjecture 1.3 with $i_1 = 1, i_2 = 3, i_3 = 4, i_4 = 6, j_1 = 2, j_2 = 3, j_3 = 4, j_4 = 5$, and

$$\begin{aligned} w &= \mathbf{5}\,3\,\mathbf{8}\,\mathbf{2}\,6\,\mathbf{4}\,7\,1 \\ w' &= 3\,\mathbf{5}\,\mathbf{8}\,\mathbf{2}\,\mathbf{4}\,6\,7\,1 \\ \pi^{(1)} = \tau' &= 3\,\mathbf{2}\,\mathbf{5}\,\mathbf{4}\,\mathbf{8}\,6\,7\,1 \\ \tau &= \mathbf{2}\,3\,\mathbf{5}\,\mathbf{4}\,6\,\mathbf{8}\,7\,1 \end{aligned}$$

(The boldface numbers are the elements in positions i_1, i_2, i_3, i_4 in w and τ and the elements in positions j_1, j_2, j_3, j_4 in w' and τ'.)

We also have that $\pi^{(2)}$ satisfies condition (1) of Conjecture 1.3 with $i_1 = 1, i_2 = 4, i_3 = 6, i_4 = 8, j_1 = 2, j_2 = 4, j_3 = 6, j_4 = 8$, and

$$\begin{aligned} w &= \mathbf{5}\,3\,8\,\mathbf{2}\,6\,\mathbf{4}\,7\,\mathbf{1} \\ w' &= 3\,\mathbf{5}\,8\,\mathbf{2}\,6\,\mathbf{4}\,7\,\mathbf{1} \\ \pi^{(2)} = \tau' &= 3\,\mathbf{2}\,8\,\mathbf{1}\,6\,\mathbf{5}\,7\,\mathbf{4} \\ \tau &= \mathbf{2}\,3\,8\,\mathbf{1}\,6\,\mathbf{5}\,7\,\mathbf{4} \end{aligned}$$

The permutation $\pi^{(3)}$ satisfies condition (1) of Conjecture 1.3 with $i_1 = 3, i_2 = 4, i_3 = 6, i_4 = 8, j_1 = 3, j_2 = 4, j_3 = 5, j_4 = 8$, and

$$\begin{aligned} w &= 5\,3\,\mathbf{8}\,\mathbf{2}\,6\,\mathbf{4}\,7\,\mathbf{1} \\ w' &= 5\,3\,\mathbf{8}\,\mathbf{2}\,\mathbf{4}\,6\,7\,\mathbf{1} \\ \pi^{(3)} = \tau' &= 5\,3\,\mathbf{2}\,\mathbf{1}\,\mathbf{8}\,6\,7\,\mathbf{4} \\ \tau &= 5\,3\,\mathbf{2}\,\mathbf{1}\,6\,\mathbf{8}\,7\,\mathbf{4} \end{aligned}$$

Finally, $\pi^{(4)}$ satisfies condition (1) of Conjecture 1.3 with $i_1 = 3, i_2 = 6, i_3 = 7, i_4 = 8, j_1 = 5, j_2 = 6, j_3 = 7, j_4 = 8$, and

$$\begin{aligned} w &= 5\,3\,\mathbf{8}\,2\,6\,\mathbf{4}\,\mathbf{7}\,\mathbf{1} \\ w' &= 5\,3\,6\,2\,\mathbf{8}\,\mathbf{4}\,\mathbf{7}\,\mathbf{1} \\ \pi^{(4)} = \tau' &= 5\,3\,6\,2\,\mathbf{4}\,\mathbf{1}\,\mathbf{8}\,\mathbf{7} \\ \tau &= 5\,3\,\mathbf{4}\,2\,6\,\mathbf{1}\,\mathbf{8}\,\mathbf{7} \end{aligned}$$

Remark 1.5. In Conjecture 1.3, given a τ' satisfying condition (1) or (2), there is in general more than one choice for the permutations w' and τ. Consider for instance the permutations w and $\tau' = \pi^{(2)}$ from Example 1.4. They satisfy condition (1) of Conjecture 1.3 with $i_1 = 1, i_2 = 2, i_3 = 6, i_4 = 8, j_1 = 1, j_2 = 4, j_3 = 6, j_4 = 8$, and

$$\begin{aligned} w &= \mathbf{5}\,\mathbf{3}\,8\,2\,6\,\mathbf{4}\,7\,\mathbf{1} \\ w' &= \mathbf{5}\,2\,8\,\mathbf{3}\,6\,\mathbf{4}\,7\,\mathbf{1} \\ \pi^{(2)} = \tau' &= \mathbf{3}\,2\,8\,\mathbf{1}\,6\,\mathbf{5}\,7\,\mathbf{4} \\ \tau &= \mathbf{3}\,\mathbf{1}\,8\,2\,6\,\mathbf{5}\,7\,\mathbf{4} \end{aligned}$$

This is a different choice of w' and τ than the one we made in Example 1.4.

In this paper we prove one direction of Conjecture 1.3, namely the sufficiency of Lakshmibai-Sandhya singularity conditions:

Theorem 1.6. *In the notation of Conjecture 1.3,*

$$\cup_\lambda X_\lambda \subseteq \operatorname{Sing} X_w \,.$$

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2 Proof of Theorem 1.6

Theorem 1.6 follows immediately from Proposition 2.1 below.

Proposition 2.1. *Let w and τ' satisfy condition (1) or (2) in Conjecture 1.3. Then $X_{\tau'} \subseteq \operatorname{Sing} X_w$.*

In the special case when $\tau' = \tau$ and $w' = w$ (in the notation of Conjecture 1.3), Proposition 2.1 was proved in [7, Lemma 3.1].

Before proving Proposition 2.1 we introduce some notation and prove a preliminary lemma. For $1 \leq i, j \leq n$, $i \neq j$, denote by $s_{ij} \in S_n$ the transposition which interchanges i and j. For $\pi \preceq \sigma \in S_n$, let $T(\sigma, \pi)$ denote the Zariski tangent space to X_w at e_π and

$$A(\sigma, \pi) \;=\; \{(i,j) \mid 1 \leq i < j \leq n \text{ and } \pi \circ s_{ij} \preceq \sigma\}\,.$$

Lakshmibai and Seshadri [8] proved that $\dim T(\sigma, \pi) = |A(\sigma, \pi)|$. Consider also the set

$$B(\sigma, \pi) \;=\; \{(i,j) \mid 1 \leq i < j \leq n,\, \pi_i < \pi_j, \text{ and } \pi \circ s_{ij} \preceq \sigma\}\,.$$

Since $\pi \circ s_{ij} \prec \pi \preceq \sigma$ for all inversions (i,j) of π, it follows that

$$A(\sigma, \pi) \;=\; \{(i,j) \mid 1 \leq i < j \leq n \text{ and } \pi_i > \pi_j\} \cup B(\sigma, \pi),$$

hence

$$|A(\sigma, \pi)| \;= l(\pi) + |B(\sigma, \pi)|.$$

Let $P = \{a_1, \ldots, a_k\}$ and $Q = \{b_1, \ldots, b_k\}$ be subsets of $\{1, \ldots, n\}$. We say that $P \leq Q$ if when the elements of P and Q are arranged in decreasing order, $a_1 \geq \cdots \geq a_k$ and $b_1 \geq \cdots \geq b_k$, we have that $a_i \leq b_i$ for $1 \leq i \leq k$. This gives a partial order on the k-element subsets of $\{1, \ldots, n\}$ for $1 \leq k \leq n$. For a sequence θ of k numbers, denote by S_θ the set of elements of θ and by $\theta_{\leq i}$, $1 \leq i \leq k$, the subsequence of θ consisting of the first i elements. In [1] Ehresmann defined the following partial order on S_n (see also [10] and [12]): If $v, w \in S_n$, then

$$v \preceq w \quad \Longleftrightarrow \quad S_{v_{\leq i}} \leq S_{w_{\leq i}} \text{ for } 1 \leq i \leq n\,.$$

It is not difficult to check that the Ehresmann order coincides with the Bruhat order. We will use this fact later in the paper.

In the following lemma we identify (i,j) and (j,i) for $1 \leq i,j \leq n$.

Lemma 2.2. *Let $\pi \prec v \preceq \sigma \in S_n$ be such that $v = \pi \circ s_{ij}$ for some $1 \leq i < j \leq n$. Define an injective map $\phi = \phi_{ij} : A(\sigma, v) \hookrightarrow \{(p,q) \mid 1 \leq p, q \leq n\}$ as follows:*

- *If $(r,t) \in A(\sigma, v)$ and $r, t \notin \{i,j\}$ or $r = i$, $t = j$, then $\phi(r,t) = (r,t)$.*

Now let $r \neq i, j$.

- *If $(r,i), (r,j) \in A(\sigma, v)$, then $\phi(r,i) = (r,i)$ and $\phi(r,j) = (r,j)$.*
- *If $(r,i) \in A(\sigma, v)$, but $(r,j) \notin A(\sigma, v)$, then*

$$\phi(r,i) = \begin{cases} (r,i), & \text{if } r < j, \pi_r < \pi_j; \\ (r,j), & \text{otherwise.} \end{cases}$$

- *If $(r,j) \in A(\sigma, v)$, but $(r,i) \notin A(\sigma, v)$, then*

$$\phi(r,j) = \begin{cases} (r,i), & \text{if } r < j, \pi_r < \pi_j; \\ (r,j), & \text{otherwise.} \end{cases}$$

Then $\operatorname{Im} \phi \subseteq A(\sigma, \pi)$.

Proof. If $(r,t) \in A(\sigma, v)$ and $r, t \notin \{i,j\}$, then $\pi \circ s_{rt} \prec v \circ s_{rt} \preceq \sigma$, so $\phi(r,t) = (r,t) \in A(\sigma, \pi)$. We also have that $\phi(i,j) = (i,j) \in A(\sigma, v), A(\sigma, \pi)$. It remains to show that for $r \neq i, j$ if both $(r,i), (r,j) \in A(\sigma, v)$, then $(r,i), (r,j) \in A(\sigma, \pi)$ and if at least one of $(r,i), (r,j)$ is in $A(\sigma, v)$, then $A(\sigma, \pi)$ contains (r,i) (resp.

(r,j)) if $r<j$ and $\pi_r<\pi_j$ (resp. $r>j$ or $\pi_r>\pi_j$). There are six possible cases and we consider each one separately:

Case 1) $r>j$, $\pi_r>\pi_j$

In this case $\pi\circ s_{rj}\prec v\circ s_{ri}$, $v\circ s_{rj}$. Therefore, if one of (r,i), $(r,j)\in A(\sigma,v)$, then $(r,j)\in A(\sigma,\pi)$. It remains to prove that if both (r,i), $(r,j)\in A(\sigma,v)$, then $(r,i)\in A(\sigma,\pi)$. Let $\alpha=\pi\circ s_{ri}$, $\beta=v\circ s_{ri}$, and $\gamma=v\circ s_{rj}$. If $k<j$, then $S_{\alpha_{\leq k}}=S_{\beta_{\leq k}}\leq S_{\sigma_{\leq k}}$. On the other hand, if $k\geq j$, then $S_{\alpha_{\leq k}}=S_{\gamma_{\leq k}}\leq S_{\sigma_{\leq k}}$. Therefore, $\alpha=\pi\circ s_{ri}\preceq w$.

Case 2) $r>j$, $\pi_r<\pi_j$

We have $\pi\circ s_{rj}\prec\pi\prec\sigma$, so $(r,j)\in A(\sigma,\pi)$. It remains to show that if both $(r,i),(r,j)\in A(\sigma,v)$, then $(r,i)\in A(\sigma,\pi)$. This follows from the fact that $\pi\circ s_{ri}\prec v\circ s_{rj}$.

Case 3) $i<r<j$, $\pi_r>\pi_j$

In this case $\pi\circ s_{rj}\prec\pi\prec w$, so $(r,j)\in A(\sigma,\pi)$. It remains to show that if $(r,i),(r,j)\in A(\sigma,v)$, then $(r,i)\in A(\sigma,\pi)$. This follows from the fact that $\pi\circ s_{ri}\prec v\circ s_{ri}$.

Case 4) $i<r<j$, $\pi_r<\pi_j$

We have that $\pi\circ s_{ri}\prec v\preceq\sigma$, hence $(r,i)\in A(\sigma,\pi)$. It remains to show that if $(r,i),(r,j)\in A(\sigma,v)$, then $(r,j)\in A(\sigma,\pi)$. This follows from the fact that $\pi\circ s_{rj}\prec v\circ s_{rj}$.

Case 5) $r<i$, $\pi_r>\pi_j$

We have that each of $\pi\circ s_{ri}$, $\pi\circ s_{rj}$, $v\circ s_{ri}$, $v\circ s_{rj}$ is smaller than v. Hence $(r,i),(r,j)\in A(\sigma,\pi),A(\sigma,v)$.

Case 6) $r<i$, $\pi_r<\pi_j$

In this case $\pi\circ s_{ri}\prec v\circ s_{ri}$, $v\circ s_{rj}$. Therefore, if one of (r,i), $(r,j)\in A(\sigma,v)$, then $(r,i)\in A(\sigma,\pi)$. It remains to prove that if both (r,i), $(r,j)\in A(\sigma,v)$, then $(r,j)\in A(\sigma,\pi)$. Let $\alpha=\pi\circ s_{rj}$, $\beta=v\circ s_{ri}$, and $\gamma=v\circ s_{rj}$. If $k<i$, then $S_{\alpha_{\leq k}}=S_{\beta_{\leq k}}\leq S_{\sigma_{\leq k}}$. On the other hand, if $k\geq i$, then $S_{\alpha_{\leq k}}=S_{\gamma_{\leq k}}\leq S_{\sigma_{\leq k}}$. Therefore, $\alpha=\pi\circ s_{rj}\preceq\sigma$. □

Proof of Proposition 2.1. Since the Borel group B acts on X_w and for $\sigma\prec w$ the closure of the orbit of e_σ is X_σ, to prove the inclusion $X_{\tau'}\subseteq\operatorname{Sing}X_w$, it will be enough to show that $e_{\tau'}$ is a singular point in X_w.

Let w and τ' satisfy conditions (1) or (2) in Conjecture 1.3. We need to show that

$$|A(w,\tau')| = \dim T(w,\tau') > \dim X_w = l(w)\,.$$

We will deal separately with conditions (1) and (2).

First assume that the permutations w and τ' satisfy condition (1) in Conjecture 1.3. If $n=4$, then $w=w'=4231$ and $\tau=\tau'=2143$, hence

$$A(w,\tau') = \{(i,j) \mid 1 \le i < j \le 4\}$$

and $|A(w,\tau')| = 6 > l(w) = 5$. Now let $n>4$. Suppose that $w_{i_1} \neq n$. Then $n \notin \{w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4}\}$, hence n is in the same position in w and τ. Since $w \succeq w' \succ \tau' \succeq \tau$ it follows that n is in the same position in w, w', τ', and τ. Therefore we can replace w, w', τ', τ by $w\setminus n$, $w'\setminus n$, $\tau'\setminus n$, $\tau\setminus n$ respectively and conclude the proof by induction on n. Thus we can assume that $w_{i_1}=n$. Similarly we can assume that $w_{i_4}=1$. The fact that $w \succeq w'$, $\tau' \succeq \tau$, and $w_{i_1}=n$ implies the following inequalities:

$$i_1 \le j_1 \text{ and } i_3 \ge j_3. \tag{1}$$

The fact that $w \succeq w'$, $\tau' \succeq \tau$, and $w_{i_4}=1$ implies that:

$$i_4 \ge j_4 \text{ and } i_2 \le j_2. \tag{2}$$

Let $v \in S_n$ be the permutation obtained from τ' by interchanging τ'_{j_1} and τ'_{j_3} as elements in τ', i.e., $v = \tau' \circ s_{j_1 j_3}$. Then $w' \succ v \succ \tau'$. The inequalities (1) and (2) imply that if $a = w_{i_2} < b = w_{i_3}$, then we can write w, w', v, τ', and τ as follows:

$$\begin{aligned}
w &= \cdots\mathbf{n}\cdots\cdots\mathbf{a}\cdots\cdots\cdots\mathbf{b}\cdots\cdots\mathbf{1}\cdots \\
w' &= \cdots\cdots\mathbf{n}\cdots\cdots\mathbf{a}\cdots\mathbf{b}\cdots\cdots\mathbf{1}\cdots\cdots \\
v &= \cdots\cdots\mathbf{n}\cdots\cdots\mathbf{1}\cdots\mathbf{a}\cdots\cdots\mathbf{b}\cdots\cdots \\
\tau' &= \cdots\cdots\mathbf{a}\cdots\cdots\mathbf{1}\cdots\mathbf{n}\cdots\cdots\mathbf{b}\cdots\cdots \\
\tau &= \cdots\mathbf{a}\cdots\cdots\mathbf{1}\cdots\cdots\cdots\mathbf{n}\cdots\cdots\mathbf{b}\cdots
\end{aligned} \tag{3}$$

(As in Example 1.4, the boldface numbers are the elements in positions i_1, i_2, i_3, i_4 in w and τ and the elements in positions j_1, j_2, j_3, j_4 in w', v, and τ'.) The only freedom in (3) is that the relative positions of the i_2th and j_1th columns can be interchanged, and also the relative positions of the i_3th and j_4th columns can be interchanged.

For example, the permutations

$$w = \mathbf{9}\,7\,5\,\mathbf{3}\,2\,8\,\mathbf{6}\,4\,\mathbf{1} \textit{ and}$$
$$\tau' = 7\,5\,\mathbf{3}\,2\,\mathbf{1}\,\mathbf{9}\,8\,\mathbf{6}\,4$$

satisfy condition (1) in Conjecture 1.3 with $i_1 = 1, i_2 = 4, i_3 = 7, i_4 = 9, j_1 = 3, j_2 = 5, j_3 = 6$, and $j_4 = 8$. Indeed, in this example

$$w' = 7\,5\,\mathbf{9}\,2\,\mathbf{3}\,\mathbf{6}\,8\,\mathbf{1}\,4,$$
$$\tau = \mathbf{3}\,7\,5\,\mathbf{1}\,2\,8\,\mathbf{9}\,4\,\mathbf{6},$$

$w_{i_1}w_{i_2}w_{i_3}w_{i_4} = w'_{j_1}w'_{j_2}w'_{j_3}w'_{j_4} = \mathbf{9\,3\,6\,1}$ has the same relative order as $4\,2\,3\,1$, and $\tau \prec \tau' \prec w' \prec w$.

We know that

$$|A(w,v)| = \dim T(w,v) \geq \dim X_w\,.$$

So to prove the desired inequality $|A(w,\tau')| > \dim X_w$ it will be enough to show that $|A(w,\tau')| > |A(w,v)|$.

In the example above

$$\begin{aligned} v =& 7\,5\,\mathbf{9}\,2\,\mathbf{1}\,\mathbf{3}\,8\,\mathbf{6}\,4, \\ B(w,\tau') =& \{(1,6),(2,6),(3,6),(4,6),(5,6),(1,7),(2,7),(3,7),(4,7), \\ & (5,7),(2,8),(3,8),(4,8),(5,8),(3,9),(4,9),(5,9)\}, \textit{ and} \\ B(w,v) =& \{(1,3),(2,3),(4,6),(5,6),(6,7),(6,8),(6,9)\}. \end{aligned}$$

Therefore in this example we have that $|A(w,\tau')| = l(\tau') + |B(w,\tau')| = 19+17 = 36$ and $|A(w,v)| = l(v) + |B(w,v)| = 20+7 = 27$, hence $|A(w,\tau')| > |A(w,v)|$.

Apply Lemma 2.2 to the triple $\tau' \prec v \prec w$. Since $(j_2,j_4) \in A(w,\tau')$, to prove the inequality $|A(w,v)| < |A(w,\tau')|$ it will be enough to show that $(j_2,j_4) \notin \operatorname{Im}\phi$, where $\phi = \phi_{j_1j_3} : A(w,v) \hookrightarrow A(w,\tau')$ is the monomorphism from Lemma 2.2. We have that $j_2, j_4 \notin \{j_1, j_3\}$, so from the definition of ϕ it follows that the only element that ϕ could possibly map to (j_2,j_4) is (j_2,j_4) itself. Thus, to prove the inequality $|A(w,v)| < |A(w,\tau')|$ it will be enough to show that $(j_2,j_4) \notin A(w,v)$.

In our example $v \circ s_{j_2 j_4} = v \circ s_{58} = 7\,5\,\mathbf{9}\,2\,\mathbf{6}\,\mathbf{3}\,8\,\mathbf{1}\,4 \not\preceq w$, hence $(5,8) \notin A(w,v)$.

Assume that $(j_2, j_4) \in A(w,v)$ and let $\xi = v \circ s_{j_2 j_4}$. Then $\xi \preceq w$. By (1) and (2) we have the inequalities $i_2 \leq j_2 < j_3 \leq i_3$, hence

$$\begin{aligned} &\text{if } S_{w_{\leq j_2}} = \{n, \alpha_2, \ldots, \alpha_{j_2-1}, \alpha_{j_2}\} \quad \text{then } S_{\tau_{\leq j_2}} = \{\alpha_2, \alpha_3, \ldots, \alpha_{j_2}, 1\} \\ &\text{if } S_{w'_{\leq j_2}} = \{n, \beta_2, \ldots, \beta_{j_2-1}, \beta_{j_2}\} \quad \text{then } S_{\tau'_{\leq j_2}} = \{\beta_2, \beta_3, \ldots, \beta_{j_2}, 1\}. \end{aligned}$$

Assume that $\alpha_2 \geq \cdots \geq \alpha_{j_2}$ and $\beta_2 \geq \cdots \geq \beta_{j_2}$. Since $w \succeq w'$ and $\tau' \succeq \tau$ it follows that $\alpha_r \geq \beta_r \geq \alpha_r$ for $2 \leq r \leq j_2$, hence $\alpha_r = \beta_r$ for $2 \leq r \leq j_2$ and $S_{w_{\leq j_2}} = S_{w'_{\leq j_2}}$. Since $\xi_r = w'_r$ for $1 \leq r \leq j_2 - 1$ and $\xi_{j_2} = w_{i_3} > w_{i_2} = w'_{j_2}$ it follows that $S_{\xi_{\leq j_2}} > S_{w'_{\leq j_2}} = S_{w_{\leq j_2}}$. This implies that $\xi \not\preceq w$, which is a contradiction.

It remains to prove that $|A(w, \tau')| > \dim X_w$ when w and τ' satisfy condition (2) in Conjecture 1.3. The proof is similar to the one for condition (1).

First, we can assume that $w_{i_2} = n$ and $w_{i_3} = 1$. The fact that $w \succeq w'$, $\tau' \succeq \tau$, and $w_{i_2} = n$ implies that:

$$i_2 \leq j_2 \text{ and } i_4 \geq j_4. \tag{4}$$

The fact that $w \succeq w'$, $\tau' \succeq \tau$, and $w_{i_3} = 1$ implies that:

$$i_3 \geq j_3 \text{ and } i_1 \leq j_1. \tag{5}$$

For example, the permutations

$$\begin{aligned} w &= \mathbf{6}\,5\,\mathbf{8}\,7\,2\,\mathbf{1}\,4\,\mathbf{3} \textit{ and} \\ \tau' &= 5\,\mathbf{1}\,7\,\mathbf{6}\,\mathbf{3}\,2\,\mathbf{8}\,4 \end{aligned}$$

satisfy condition (2) in Conjecture 1.3 with $i_1 = 1, i_2 = 3, i_3 = 6, i_4 = 8, j_1 = 2, j_2 = 4, j_3 = 5$, and $j_4 = 7$. Indeed, in this example

$$\begin{aligned} w' &= 5\,\mathbf{6}\,7\,\mathbf{8}\,\mathbf{1}\,2\,\mathbf{3}\,4, \\ \tau &= \mathbf{1}\,5\,\mathbf{6}\,7\,2\,\mathbf{3}\,4\,\mathbf{8}, \end{aligned}$$

$w_{i_1} w_{i_2} w_{i_3} w_{i_4} = w'_{j_1} w'_{j_2} w'_{j_3} w'_{j_4} = \mathbf{6\,8\,1\,3}$ has the same relative order as $3\,4\,1\,2$, and $\tau \prec \tau' \prec w' \prec w$.

Let $v \in S_n$ be the permutation obtained from τ' by interchanging τ'_{j_1} and τ'_{j_2} as elements in τ', i.e., $v = \tau' \circ s_{j_1 j_2}$. Then $w' \succ v \succ \tau'$. As in the case of condition (1), to prove that $|A(w,\tau')| > \dim X_w$ it will be enough to show that $|A(w,\tau')| > |A(w,v)|$.

In the example above

$$\begin{aligned} v =& 5\,\mathbf{6}\,7\,\mathbf{1}\,\mathbf{3}\,2\,\mathbf{8}\,4, \\ B(w,\tau') =& \{(1,4),(2,4),(2,5),(2,6),(3,7),(4,7),(5,7),(5,8)\}, \textit{ and} \\ B(w,v) =& \{(1,2),(4,5),(4,6),(3,7),(5,7),(5,8)\}. \end{aligned}$$

Therefore in this example we have that $|A(w,\tau')| = l(\tau') + |B(w,\tau')| = 13+8 = 21$ and $|A(w,v)| = l(v) + |B(w,v)| = 14+6 = 20$, hence $|A(w,\tau')| > |A(w,v)|$.

We have that $(j_2,j_4) \in A(w,\tau')$. We will prove that $|A(w,v)| < |A(w,\tau')|$ by showing that $(j_2,j_4) \notin \operatorname{Im}\phi$, where $\phi = \phi_{j_1 j_2}$ is the monomorphism from Lemma 2.2 applied to the triple $\tau' \prec v \prec w$. Since $j_4 \neq j_1, j_2$, it follows from the definition of ϕ that if $(j_2,j_4) \in \operatorname{Im}\phi$, then $\phi^{-1}(j_2,j_4)$ is either (j_1,j_4) or (j_2,j_4). We will prove that this is impossible by showing that $(j_1,j_4),(j_2,j_4) \notin A(w,v)$.

In our example $v \circ s_{j_1 j_4} = v \circ s_{27} = 5\,\mathbf{8}\,7\,\mathbf{1}\,\mathbf{3}\,2\,\mathbf{6}\,4 \not\preceq w$ and $v \circ s_{j_2 j_4} = v \circ s_{47} = 5\,\mathbf{6}\,7\,\mathbf{8}\,\mathbf{3}\,2\,\mathbf{1}\,4 \not\preceq w$, hence $(2,7),(4,7) \notin A(w,v)$.

Assume first that $(j_1,j_4) \in A(w,v)$ and let $\xi = v \circ s_{j_1 j_4}$. Then $\xi \preceq w$, so in particular $i_2 \leq j_1$. By (5) we also have that $i_3 \geq j_3$. Therefore,

$$\begin{aligned} \text{if } S_{w_{\leq j_1}} &= \{n, \alpha_2, \ldots, \alpha_{j_1-1}, \alpha_{j_1}\} \quad \text{then } S_{\tau_{\leq j_1}} = \{\alpha_2, \alpha_3, \ldots, \alpha_{j_1}, 1\} \\ \text{if } S_{\xi_{\leq j_1}} &= \{n, \beta_2, \ldots, \beta_{j_1-1}, \beta_{j_1}\} \quad \text{then } S_{\tau'_{\leq j_1}} = \{\beta_2, \beta_3, \ldots, \beta_{j_1}, 1\}. \end{aligned}$$

Since $w \succeq \xi$ and $\tau' \succeq \tau$ it follows that $\alpha_r \geq \beta_r \geq \alpha_r$ for $2 \leq r \leq j_1$, hence $S_{w_{\leq j_1}} = S_{\xi_{\leq j_1}}$. But $w_{i_1} \in S_{w_{\leq j_1}}$, whereas $w_{i_1} = \xi_{j_4} \notin S_{\xi_{\leq j_1}}$, which is a contradiction. Therefore $(j_1,j_4) \notin A(w,v)$.

Now assume that $(j_2,j_4) \in A(w,v)$ and let $\eta = v \circ s_{j_2 j_4}$. Then $\eta \preceq w$, so in particular $i_3 \geq j_4$. By (4) we also have that $i_2 \leq j_2$. This implies that

$$\begin{aligned} \text{if } S_{w_{\leq j_3}} &= \{n, \alpha_2, \ldots, \alpha_{j_3-1}, \alpha_{j_3}\} \quad \text{then } S_{\tau_{\leq j_3}} = \{\alpha_2, \alpha_3, \ldots, \alpha_{j_3}, 1\} \\ \text{if } S_{\eta_{\leq j_3}} &= \{n, \beta_2, \ldots, \beta_{j_3-1}, \beta_{j_3}\} \quad \text{then } S_{\tau'_{\leq j_3}} = \{\beta_2, \beta_3, \ldots, \beta_{j_3}, 1\}. \end{aligned}$$

Since $w \succeq \eta$ and $\tau' \succeq \tau$ it follows that $\alpha_r \geq \beta_r \geq \alpha_r$ for $2 \leq r \leq j_3$, hence $S_{w_{\leq j_3}} = S_{\eta_{\leq j_3}}$. But $w_{i_4} \notin S_{w_{\leq j_3}}$, while $w_{i_4} = \eta_{j_3} \in S_{\eta_{\leq j_3}}$, which is a contradiction. Therefore $(j_2,j_4) \notin A(w,v)$.

This completes the proof of Proposition 2.1. □

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