Improving wave-equation fidelity of Gaussian beams by solving the complex eikonal equation

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SUMMARY

Gaussian beams are a well-known high-frequency wavefield approximation. A more accurate representation can be obtained by the complex eikonal equation. We propose a constructive algorithm for solving the complex eikonal equation. By re-writing the complex traveltime as background real and imaginary parts and their respective perturbations, we arrive at an update scheme that aims at solving the complex eikonal equation iteratively. The initial prior may come from the Gaussian beam approximation computed by dynamic ray tracing. Proper boundary conditions can ensure correct update directions. The result embraces complete details of the velocity model and therefore can help enhancing accuracy of Gaussian-beam migration and other related applications.

INTRODUCTION

It is well-known that by introducing complex attributes into traveltimes, one could enhance the accuracy of ray methods (Popov, 2002). In the field of seismic imaging, the method of Gaussian beams is an example of this idea. One could represent seismic rays using complex traveltimes and reach reasonable approximations even around caustics. Elegant works of Hill (1990, 2001) and others (Costa et al., 1899; Hale, 1992; Hale and Witte, 1992; Alkhalifah, 1995; Nowack, 2003; Gray, 2005, 2007; Gray and Bleistein, 2009) have shown the power of such techniques.

The dynamic ray tracing is a standard method for approximating complex eikonal equation along its real characteristics (Choudhary and Felsen, 1974; Popov, 1982; Cerveny, 1982; Cerveny et al., 1982; Cerveny, 2001; Bleistein, 2009). By integrating the imaginary part along real rays, one can construct a solution of complex eikonal that is asymptotically accurate near the central ray. Although dynamic ray tracing takes into account second derivatives of velocity, the integration is carried out only along the central ray and thus may not accurately reflect velocity details aside. Shooting dense central rays could overcome this weakness but brings a computational burden in practice, which undermines an important advantage of Gaussian beams. Another approach is higher-order beams (Tanushev, 2008), which require higher-order derivatives and corresponding high-order smoothness of the velocity background.

A way to construct more accurate beams is to solve the complex eikonal equation. Mathematically, splitting the traveltime into real and imaginary parts and inserting them into the eikonal equation leads to a system of second-order nonlinear equations of a mixed elliptic-parabolic type (Magnanini and Talenti, 1999). Unfortunately, the solution does not propagate along rays and requires expensive numerical methods. In 2D case, one can apply a Backlund transformation between the real and imaginary parts and decouple the system into semilinear second-order partial differential equations with polynomial nonlinearities. Klimes (2009a,b) suggests viewing the system as two Hamilton-Jacobian equations. He uses a method similar to wavefront tracing. However, the selection of surfaces constitutes a difficult task in order to respect the multi-valued nature of action functions. Furthermore, the finite-difference stencil meets problem when approaching caustics.

In this paper, we propose to solve the complex eikonal by turning it into a nonlinear least-square optimization problem and iteratively updating a prior solution with perturbations generated from its linearized version. Finite-difference solutions of the traditional real-valued eikonal equation have been extensively studied (Sethian, 1999). Aldridge (1994), Fomel (1997), and Franklin and Harris (2001) introduced the linearized eikonal equation and took advantage of an implicit finite-difference scheme with superior stability and accuracy properties. Here we expand the same idea to complex eikonal while the resulting problem formulation is quite different. As typical for all similar methods, our approach relies on the initial prior. It also provides a flexibility in handling multi-pathing.

LINEARIZING COMPLEX EIKONAL EQUATION

The eikonal equation, as derived from the WKBJ method (Chapman, 2004), has the form

$$\nabla T \cdot \nabla T = S^2, \tag{1}$$

where $T(x,y,z)$ and $S(x,y,z)$ are traveltime and slowness in physical space, respectively. One can get complex-valued eikonal from (1) by assigning imaginary part to the traveltime, as follows:

$$T = R + i \cdot I. \tag{2}$$

Now we have instead of one equation (1) a system of two partial differential equations governing the real and imaginary traveltimes:

$$\nabla R \cdot \nabla R = S^2 + \nabla I \cdot \nabla I, \tag{3}$$

$$\nabla R \cdot \nabla I = 0. \tag{4}$$

Equation (4) states that the gradients of the real and imaginary parts should be orthogonal. Equation (3) is non-linear, which leads to multiple branches of the solution. Similar to the real-valued eikonal, complex eikonal also suffers from multi-pathing. In order to solve equations (3) and (4), we define a new real parameter $w(x,y,z)$ as follows:

$$\begin{cases} \nabla I \cdot \nabla I &= w^2 \\ \nabla R \cdot \nabla R &= S^2 + w^2 \end{cases} \tag{5}$$

One may view w as slowness for the imaginary part. By doing so, we split the real and imaginary parts and attribute their propagation to their own slowness fields. Equation (3) is satisfied immediately, while equation (4) can provide an objective function for estimating w. We choose to minimize $\nabla R \cdot \nabla I$ in the least-square sense, i.e minimize $|\nabla R \cdot \nabla I|^2$ with respect to w. Denoting $\nabla R \cdot \nabla I$ as $F[w]$, the Gauss-Newton method correspondingly reads (k is index of iteration):

$$\delta w = w_{k+1} - w_k = -[(\nabla_w F)^T \cdot \nabla_w F]^{-1} (\nabla_w F)^T F[w_k] \quad (6)$$

To define $\nabla_w F$, we express w as a sum of some background and small perturbation, i.e $w = w_0 + \delta w$. Similarly, we have $R = R_0 + \delta R$ and $I = I_0 + \delta I$. Inserting these equations into (5) and neglecting second-order terms, we finally arrive at

$$\begin{cases} \nabla I_0 \cdot \nabla(\delta I) & = & w_0 \delta w \\ \nabla R_0 \cdot \nabla(\delta R) & = & w_0 \delta w \end{cases} \quad (7)$$

Notice that S is treated as known constant and is thus canceled. From (4) we have,

$$\delta F[w] = \nabla_w F \delta w = \nabla R_0 \cdot \nabla(\delta I) + \nabla I_0 \cdot \nabla(\delta R) \quad (8)$$

Equation (7) indicates linear relations between δw and δR, δI. If we denote linear operators as $L_R = \nabla R_0 \cdot \nabla$ and $L_I = \nabla I_0 \cdot \nabla$, then

$$\begin{cases} \delta I & = & L_I^{-1} w_0 \delta w \\ \delta R & = & L_R^{-1} w_0 \delta w \end{cases}, \quad (9)$$

and equation (8) takes the form

$$\nabla_w F \delta w = [L_I \cdot L_R^{-1} + L_R \cdot L_I^{-1}] w_k \delta w \quad (10)$$

and completes the definition of $\nabla_w F$. This construction corresponds to the adjoint-state method in traveltime tomography (Sei and Symes, 1994; Taillandier et al., 2009).

ALGORITHM

Based on equations (6) and (10), we propose the algorithm as follows:

1. Start with an initial traveltime field. It should provide both real and imaginary parts R_0 and I_0. Outputs of Gaussian beam computed by dynamic ray tracing could provide a good initial guess considering their asymptotic accuracy. It could also be a previous computation or (for simple models) the result of an approximate analytic evaluation.
2. Calculate the initial slowness w_0 according to (5). Both prior real and imaginary traveltimes can provide w_0, but the imaginary part is preferable since it guarantees $w_0^2 \geq 0$.
3. Solve linear system in equation (6) for δw. One may use various numerical methods, such as the conjugate-gradient algorithm (Hestenes and Stiefel, 1952).
4. Update w with δw and re-compute the real and imaginary traveltimes as in (5) using an eikonal solver.
5. Repeat the loop.

Linear operators in equation (10) can be computed effectively and accurately with an explicit upwind finite-difference scheme (Franklin and Harris, 2001).

Proper boundary conditions (BCs) are critical for ensuring correct convergence direction. They also directly control the beam width that one expects in the output. At least two BCs are feasible in this application. One is to impose zero imaginary traveltime along the central ray. The other requires an exact solution at the surface. A construction for such analytical surface solutions is discussed in the next section.

SYNTHETIC TESTS

The first test is with a simple constant velocity background $v(\mathbf{x}) = v_0 = 1.5 km/s$. By taking an exact solution of the real-valued eikonal equation for a point source $\mathbf{y} = (y_1, y_2, y_3)$ and moving the source point to the complex plane, we construct an exact solution of the complex eikonal equation.

$$T(\mathbf{x}) = \frac{\sqrt{(x_1 - y_1 - is)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} + is}{v_0} \quad (11)$$

Equation (11) corresponds to the phase of the family of exact constant-velocity wave-equation solutions (Keller and Streifer, 1971; Deschamps, 1971; Felsen, 1984; Wu, 1985; Kiselev and Perel, 2002). The Gaussian beam in this case is an exact solution of the paraxial complex eikonal equation, as follows:

$$T(\mathbf{x}) \approx \frac{x_1 - y_1}{V_0} + \frac{(x_2 - y_2)^2 + (x_3 - y_3)^2}{2V_0(x_1 - y_1 - is)} \quad (12)$$

We use the Gaussian beam (12) as initial guess for our nonlinear inversion and test if we can converge to the exact solution (11). Inversion is converged after only 4 iterations. Results are shown in Figures 1 (top two rows) and 2. There is only a small improvement on beam after update. According to equations (11) and (12), Gaussian beam is approximating a hyperbola with a parabola. It fits exact solution quite well around the apex, i.e. the central ray. Although errors are significant far away from the central way, the growing imaginary parts annihilate the amplitude.

We can expect that, in more complicated media, the update can recover wave-fields that Gaussian beam fails to approximate accurately. The second example justifies this idea. The model contains a Gaussian anomaly that is intentionally put away from the central ray (at location $x_2 = 0$ and propagates vertically downward) so that its influence is very small at central ray. We thus assume that central ray is not perturbed. Then we apply dynamic ray tracing with the standard 4th order Runge-Kutta method to calculate Gaussian beam and use it again as initial guess for inversion. Dynamic ray tracing is expected to feel neighboring medium by taking into account second-derivatives of the velocity in the direction perpendicular to the central ray. However as shown in Figure 4, Gaussian beam fails to properly account for the anomaly, while after update, beam in anomaly area is clearly upgraded. Moreover, in theory of Gaussian beam, we restrict the beam shape to be symmetric around the central ray. Therefore it is impossible for a Gaussian beam to capture such one-sided anomaly as in this example. Figure 1 (bottom two rows) compares real and imaginary traveltimes before and after update.

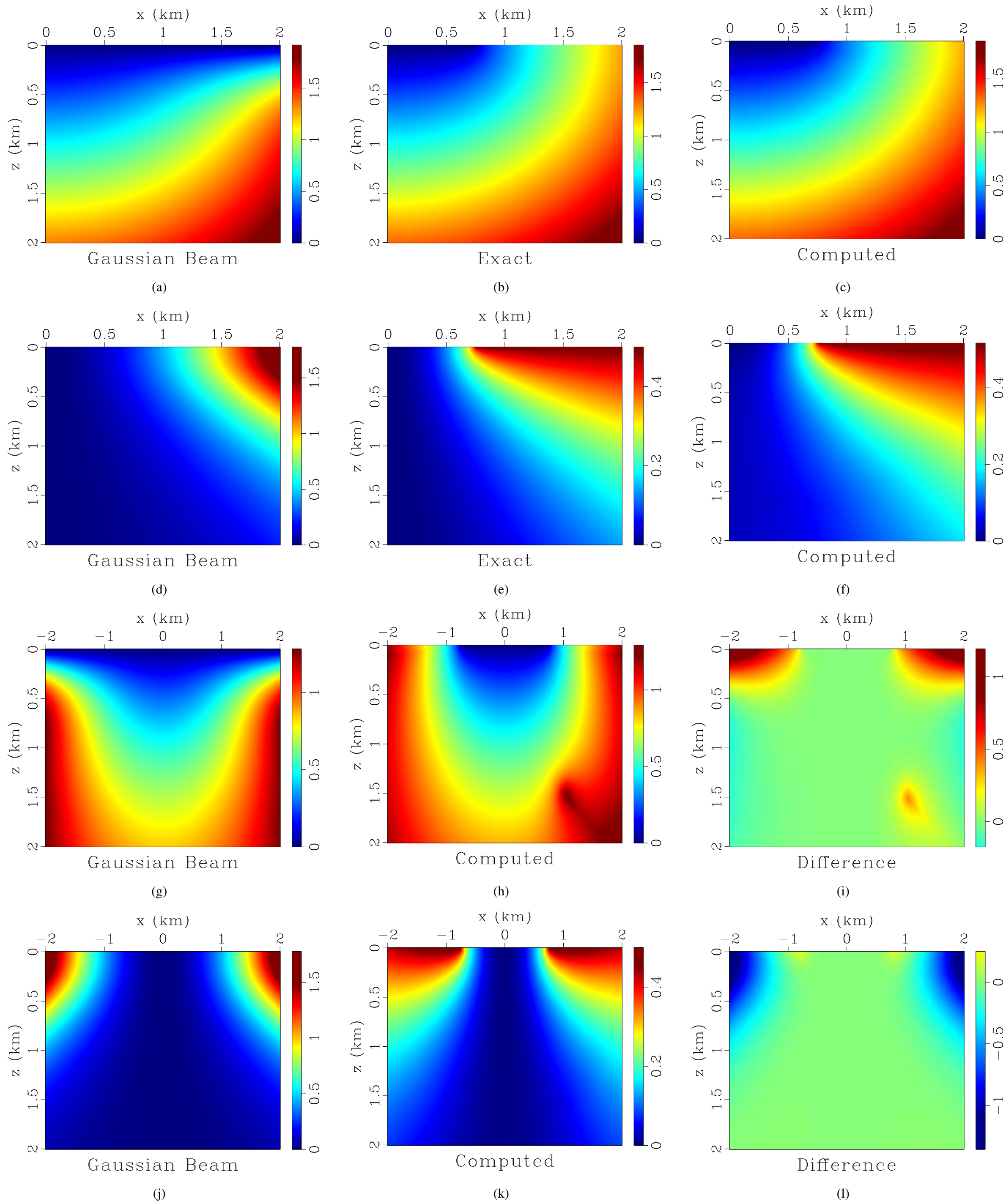

Figure 1: First row: constant velocity example real part traveltime. Second row: constant velocity example imaginary part traveltime. Third row: Gaussian anomaly example real part traveltime. Fourth row: Gaussian anomaly example imaginary part traveltime.

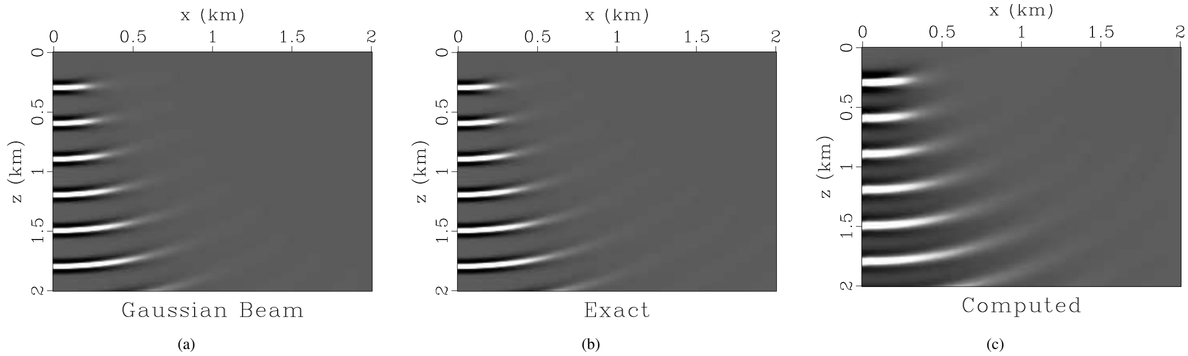

Figure 2: Example waveforms for constant-velocity vertical beam.

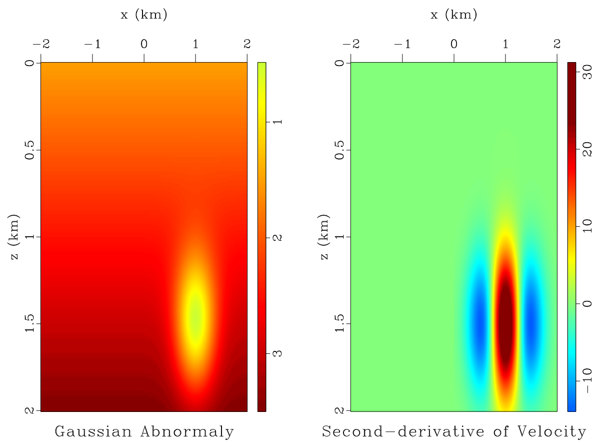

Figure 3: The model with a Gaussian anomaly and its second derivative along x_2 direction (perpendicular to central ray).

CONCLUSIONS

We propose a new approach to solving the complex eikonal equation. The original complex eikonal is split into real and imaginary parts by introducing a pseudo-slowness. Gauss-Newton iteration is then established to minimize the product of gradients of the real and imaginary traveltimes, as required by the complex eikonal equation. Boundary conditions are essential for iterative convergence to the correct solution.

Preliminary numerical tests on simple velocity models show good accuracy of the proposed method and suggest its potential application in areas such as Gaussian beam modeling and migration. The updated real and imaginary traveltimes incorporate more velocity details. Therefore, we expect our method to enhance the wave-equation fidelity of Gaussian beams. For computational efficiency, the update can be limited to a small region around the central ray that contains significant wave amplitudes. There is a limitation associated with using first-arrival eikonal solvers for updating real and imaginary parts of the complex traveltime. To overcome this limitation, one may need to adopt phase-space extensions of the eikonal equation.

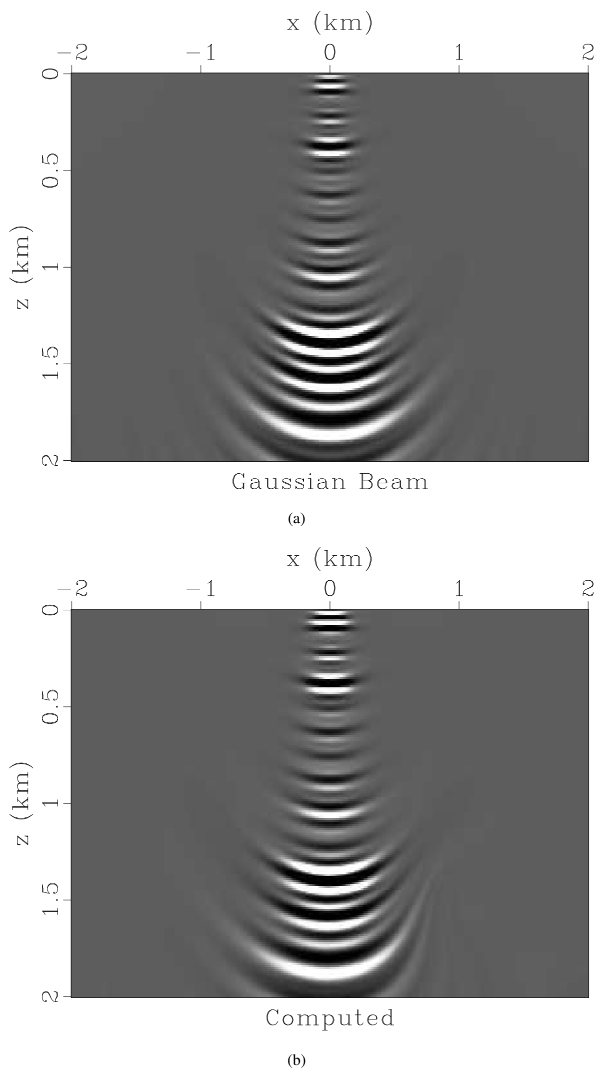

Figure 4: Model with a Gaussian anomaly wave-field (a) before and (b) after update.

EDITED REFERENCES

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