Some Examples of Limits of Kleinian Groups

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August 21, 2007

The goal of this lecture will be to exhibit the thick-thin decomposition for a collection of examples of Kleinian groups. In this lecture, we’ll demonstrate how geometric limits, as opposed to algebraic limits, can be a good tool to do this.

The limiting phenomena that we would like to in this lecture discuss are the following:

1. (Jørgensen) A sequence of representation $\rho_n : \mathbb{Z} \to \text{Isom}^+(\mathbb{H}^3)$, where $\{\rho_n(\mathbb{Z})\}$ converges geometrically to a group isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

2. (Kerchoff-Thurston) A sequence of quasi-Fuchsian groups given by iteration of a Dehn twist on a Bers slice, such that $Q(X, T^n X)$ has no parabolics but converges geometrically to a manifold with a rank two cusp.

3. (Jørgensen-Thurston-Bonahon-Otal) A sequence of groups $Q_n$ that converges geometrically to group whose quotient manifold has infinitely many cusps. This is done by doing higher and higher Dehn twists on a sequence of curves.

In each of these examples, the boundaries of the thin parts remain controlled, so the thick part of the manifold stabilizes. In the next example this is not the case.

4. A sequence of groups $Q(X, \phi^n X)$, where $\phi$ is a “partially pseudo-Anosov” mapping class, i.e. when $\phi$ restricted to some component or components is a pseudo-Anosov, and when restricted to others is the identity map.

![Diagram]

As an example of this last type, suppose $\phi$ were given by $\tau^{-1}_\alpha \circ \tau_\beta$, where $\alpha$ and $\beta$ are the
curves shown above. Then $\phi$ is a pseudo-Anosov when restricted to the one-holed torus on the right of the above diagram.

In this case, the convex core boundaries on one side of the manifold stay close together, as there is a “figure eight” homotopy between them. To push curves through the other side of the manifold, however, takes a very long time, and we see a large Margulis tube appearing.

![Schematic Picture](image)

Because the restriction of $\phi$ to the right hand side of the surface is pseudo-Anosov, the right half of the manifold looks increasingly like a cyclic cover of a manifold fibering over the circle with punctured torus fiber. In the approximates, we therefore see a manifold whose geometric structure on one side is roughly periodic.
Jørgensens example- As a first example, we return to the sequence of representations
\( \rho_n : \mathbb{Z} \to \text{PSL}_2(\mathbb{C}) \) defined by
\[ \rho_n(1) = \begin{bmatrix} e^{\omega_n} & n \sinh(\omega_n) \\ 0 & e^{-\omega_n} \end{bmatrix}, \]
where \( \omega_n = \frac{1}{n^2} + \frac{\pi i}{n} \).

To understand the geometric limit of these representations, we fix attention on a single point \( x \). For any \( n \), the loxodromic element \( \rho_n(1) \) corresponds to twisting \( x \) around a cone whose axis is the axis of \( \rho_n(1) \). As \( n \) increases, the axis of \( \rho_n(1) \) moves further and further from \( x \). The following illustration shows these cones for \( m > n \).

\( \rho_n(1) \) and \( \rho_n(n) \) translate \( x \) by roughly the same amount, but in different directions along the cone. As \( n \) increases, the cone through \( x \) becomes very flat, and in the limit we see a horosphere through \( x \). This is clearly visible in the ball model, in which the regular neighborhood of the loxodromic axis is a banana rather than a cone, and one sees the endpoints of its axis coming closer and closer together.

By the definition of the geometric limit, both \( \lim_{n \to \infty} \rho_n(1) \) and \( \lim_{n \to \infty} \rho_n(n) \) must be in the geometric limit, so the geometric limit has two generators.

Note that \( \lim_{n \to \infty} \rho_n(n) \) is not in the algebraic limit, as the algebraic limit consists of limits
of representations of fixed elements. The algebraic limit is generated by the limits of the
generators, so in this case is a cyclic group generated by the single parabolic \( \lim_{n \to \infty} \rho_n(1) \).

While the algebraic limit is isomorphic to the original group, and the geometric limit is not,
though the quotient manifold of the geometric limit looks much more like the approximates
than the quotient manifold of the algebraic limit.

**The Kerhoff-Thurston Example**-

In this example, we look at a sequence of quasi-Fuchsian groups whose convex core bound-
daries have controlled geometry, but whose geometric limit develops a rank-two cusp in its
interior.

Let \( X \) be a point in the Teichmüller space of genus two surfaces, and \( \tau \) the element of the
modular group given by Dehn twisting around the curve \( \delta \) shown below.

Consider the sequence of quasi-Fuchsian three manifolds \( Q(X, \tau^n X) \) given by simultaneous
uniformization of \( X \) and \( \tau^n X \). This sequence sits in the Bers slice based at \( X \), which is
precompact in the algebraic topology. As a result, by passing to a subsequence we can
ensure that \( Q(X, \tau^n X) \) converges to \( Q_\infty = \mathbb{H}^3 / \Gamma_A \).

Let \( \Gamma_G \) be the geometric limit of the
groups \( Q(X, \tau^n X) \), \( M_G \) the quotient manifold.

**Theorem:** (Kerckhoff-Thurston) \( M_G \cong S \times \mathbb{R} - \delta \times 0 \)

One sees a rank two cusp appear in the limit coming from a tubular neighborhood of the
curve \( \delta \). \( \pi_1(N(\delta)) \cong \mathbb{Z} \oplus \mathbb{Z} \), and this gives a rank two parabolic subgroup of the geometric
limit.

\(^1\)Actually, this sequence converges without passing to subsequence.
To prove this theorem, we fix a set of generators for the fundamental group of the surface and consider how $\tau$ acts on these generators. We choose a basepoint on one side of the curve $\delta$, and generators as shown below.

As an automorphism of the fundamental group, $\tau_*$ fixes $\alpha_1$ and $\beta_1$, but conjugates $\alpha_2$ and $\beta_2$ by $\delta$. As $Q(X, \tau^n X) \to Q_\infty = \mathbb{H}^3/\Gamma_A$, we can choose representations $\rho_n : \pi_1(S) \to \text{Isom}^+(\mathbb{H}^3)$ with $Q(X, \tau^n X) = \mathbb{H}^3/\rho_n(\pi_1(S))$ and $\rho_n(\pi_1(S)) \to \Gamma_A$ as $n$ goes to infinity.

A key trick in analyzing limits of such iterations is to notice that by precomposing the representation by an automorphism of the group, one does not change the image Kleinian group. If $\rho'_n = \rho_n \circ \tau_*^{-1}$, then the images of $\rho_n$ and $\rho'_n$ are the same quasi-Fuchsian group. This sequence is given by remarking the initial surface using $\tau$. We claim that $\rho'_n$ is also precompact, so again we can find a sequence $\rho'_n$ that converges algebraically.

In changing from the sequence $\rho_n$ to the sequence $\rho'_n$, we have shifted our attention from one side of the quasi-Fuchsian manifold to the other, however as far as the images of these representations are concerned we have done nothing. We are looking at the same group, we have simply changed our perspective on which end is twisted. As the sequence of groups is the same, their geometric limits are the same group $\Gamma_G$.

We already have a list of elements that must be in the group $\Gamma_G$, namely the limits of $\rho_n(\delta)$

- $\rho_n(1) = \rho'_n(\alpha_1)$
- $\rho_n(\beta_1) = \rho'_n(\beta_1)$
- $\rho_n(\alpha_2)$, $\rho_n(\beta_2)$
- $\rho'_n(\alpha_2) = \rho_n(\delta^n)\rho_n(\alpha_2)\rho_n(\delta^{-n})$, $\rho'_n(\beta_2) = \rho_n(\delta^n)\rho_n(\beta_2)\rho_n(\delta^{-n})$

We claim that $\rho_n(\delta^n)$ also converges to some $\delta \in \text{Isom}(\mathbb{H}^3)$. To see this, notice that $\rho_n(\delta^n)$ sends the fixed points of $\rho_n(\alpha_2)$ to the fixed points of $\rho'_n(\alpha_2)$, as if $\rho_n(\alpha_2)x = x$, then $\rho'_n(\alpha_2)(\rho_n(\delta^n)x) = (\rho'_n(\alpha_2)\rho_n(\delta^n))x = (\rho_n(\delta^n)\rho_n(\alpha_2))x = \rho_n(\delta^n)x$. In fact, the same argument shows that for any word $w \in \langle \alpha_2, \beta_2 \rangle$, $\rho_n(\delta^n)$ sends the set of fixed points of $\rho_n(w)$ to the set of fixed points for $\rho'_n(w)$. As $\alpha_2$ and $\beta_2$ generate a rank-2 free group, there are infinitely many boundary points that are fixed by words $w \in \langle \alpha_2, \beta_2 \rangle$. We can therefore pick three words $w_1, w_2$ and $w_3$ and distinct points on the sphere at infinity $x_1^n, x_2^n$ and $x_3^n$ such that $\rho_n(w_1)x_1^n = x_1^n, \rho_n(w_2)x_2^n = x_2^n$ and $\rho_n(w_3)x_3^n = x_3^n$. As $n$ goes to infinity,
\[ x^n \text{ converges to a fixed point } x^\infty_i \text{ of } \rho_\infty(w_i) \text{ and } \rho_n(\delta^n)x^n_i \text{ converges to a fixed point } y_i \text{ of } \rho_\infty^i(w_i). \] Thus \( \lim_{n \to \infty} \rho_n(\delta^n)x^\infty_i = y_i. \) This shows that \( \rho_n(\delta^n) \) converges to the unique Möbius transformation \( \delta \) that sends \( x^\infty_1 \mapsto y_1, x^\infty_2 \mapsto y_2 \) and \( x^\infty_3 \mapsto y_3. \)

As \( \rho_n(\delta^n) \to \delta, \delta \) is in \( \Gamma_G \) by the definition of the geometric limit. We claim that \( \delta \) is not in the algebraic limit. Suppose it were, i.e. suppose there exists an element \( g \) such that \( \lim_{n \to \infty} \rho_n(g) = \delta. \) Then \( \rho_n(g) \circ \rho_n(\delta^{-n}) = \rho_n(g\delta^{-n}) \to \text{id}. \) By Gromov’s compactness theorem (see exercises), the geometric limit is a discrete group, so there is some lower bound \( \epsilon \) on the injectivity radius of all elements of \( \Gamma_G. \) Thus if \( \epsilon_n \) is the injectivity radius of \( \rho_n(\pi_1(S)), \) then \( \rho_n(\pi_1(S)) \to \Gamma_G \) implies that \( \epsilon_n \to \epsilon. \) In particular, for large enough \( n, \) the injectivity radius of elements in \( \rho_n(\pi_1(S)) \) is bounded below by \( \epsilon/2. \) As \( \rho_n(g\delta^{-n}) \to \text{id}, \) for sufficiently large \( n \) \( \rho_n(g\delta^{-n}) \) has injectivity radius below \( \epsilon/2, \) and therefore equals the identity. The representations \( \rho_n \) are faithful, however, so this is a contradiction.

We can also show \( \langle \delta, \rho_\infty(\delta) \rangle \cong \mathbb{Z} \oplus \mathbb{Z}. \) The first thing to check is that these two elements commute. \( \rho_n(\delta^n) \) and \( \rho_n(\delta) \) commute for all \( n, \) so the limit of their commutator is the identity. This shows that \( \langle \delta, \rho_\infty(\delta) \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \) or \( \langle \delta, \rho_\infty(\delta) \rangle \cong \mathbb{Z}. \) If the latter is the case, then \( \delta^i = \rho_\infty(\delta)^j \) for some \( i \) and \( j, \) so \( \rho_n(\delta^ni)\rho_n(\delta^{-j}) = \rho_n(\delta^{ni-j}) \to \text{id}. \) Applying the same argument as before, we have that \( \rho_n(\delta^{ni-j}) \) is equal to the identity for sufficiently large \( n, \) which again contradicts the faithfulness of the representation.

We now know that we have a rank two free abelian subgroup, which must correspond to a cusp in a hyperbolic manifold. Because \( \delta \) is homotopic to the core curve of this cusp in the approximates, the same is true in the limit, so the quotient of \( \mathbb{H}^3 \) by the group generated by the elements we have found so far is homeomorphic to \( S \times \mathbb{R} - \delta \times \{0\}. \) One can use the theory of Klein-Maskit combination to show that there are no other elements in the geometric limit, but we won’t give the details of that here.

**An Example where the Geometric Limit is not Finitely Generated**

Let \( \eta \) and \( \delta \) be a pair of filling curves, i.e. curves \( \eta \) and \( \delta \) such that \( S - \{\eta \cup \delta\} \) is homeomorphic to a disjoint union of disks.

![Diagram](image_url)

The previous example dealt with simultaneous uniformization of a surface and a re-marking of that surface by a Dehn twist about a single curve. In this example, we will Dehn twist
about a sequence of curves. Consider the sequence of manifolds

\[ Q(X, \tau_δτ_ηX), Q(X, \tau_δ^2τ_η^2 τ_η τ_δ^2X), ..., Q(X, \tau_δ^nτ_η^n τ_η \cdots τ_δ^nτ_η^nX), ... \]

These have an algebraic limit \( Q_∞ \) by the compactness of the closure of the Bers slice, but this algebraic limit gives very little information about the approximates. As far as the algebraic limit is concerned only the final \( t_δ^n \) is important. This is because the image of any curve under the mapping class \( \tau_δ^nτ_η^n \cdots τ_δ^nτ_η^nX \) will look like a curve that is highly twisted about \( δ \). The algebraic limit of this sequence will look like the algebraic limit in the Kerckhoff-Thurston example, so we see a cusp in the limit corresponding to \( δ \).

By performing the same remarking trick as in the previous example, however, we can bring different twists in the sequence into focus, e.g. the algebraic limit of the sequence

\[ Q(\tau_δ^{-1}X, \tau_ηX), Q(\tau_δ^{-2}X, \tau_η^2 τ_δ^2τ_η^2X), ..., Q(\tau_δ^{-n}nX, \tau_η^n τ_δ^n τ_η \cdots τ_δ^nτ_η^nX), ... \]

will have a cusp corresponding to a different curve. Iterating this remarking trick, we can see a whole sequence of cusps. Just like in the previous example, the topology of our surfaces is \( S \times \mathbb{R} \) after filling in some deleted curves, however now this sequence of curves goes off to infinity.

Limits with missing subsurfaces-

We will now see what happens when we iterate a reducible mapping class. Let \( φ \) be a mapping class on a genus two surface whose restriction to one half of the surface is a pseudo-Anosov and whose restriction to the other is the identity. The sequence of manifolds \( Q(X, φ^n(X)) \) converges algebraically to a group \( Q_∞ \) that is partially degenerate, where some of the ends have short curves exiting them and others don’t.

If we look at \( Q(X, φ^n(X)) \) for large \( n \), we see a bounded homotopy on one side of the manifold, as the curves \( α_1 \) and \( β_1 \) on one convex core boundary are homotopic to their representatives on the other side of the convex core. Any homotopy between curves on the pseudo-Anosov side of the manifold, however, takes longer and longer to occur. One way to see this is to notice that the bounded length curves on one side of the manifold are \( α_2 \) and \( β_2 \), and the bounded length curves on the other side are \( φ^n(α_2) \) and \( φ^n(β_2) \). As \( φ^n(α_2) \)
and $\phi^n(\beta_2)$ intersect $\alpha_2$ and $\beta_2$ a lot, the collar lemma gives that $\alpha_2$ and $\beta_2$ must be very long on the side where $\phi^n(\alpha_2)$ and $\phi^n(\beta_2)$ have bounded length.

The geometric limit of this sequence is homeomorphic to $S \times \mathbb{R} - S'$, where, $S'$ is the subsurface on which $\phi$ acts as a pseudo-Anosov.

Geometrically what we see in the approximates is a Margulis tube appearing which gives one half of the manifold room to spread out, as was described at the beginning of this talk. This Margulis tube is different from those in the previous examples, because its boundary does not have controlled geometry. For large $n$, the boundary of the thin part of the manifold becomes very large.

Because $\phi$ restricted to $S'$ is a pseudo-Anosov, the side of the manifold corresponding to $S'$ looks more and more like the $\mathbb{Z}$ cover of a manifold fibering over the circle with fiber $S'$. The limit therefore has a periodic geometric structure on this side. A more faithful geometric picture of this manifold is shown below.

As the above picture shows, the geometric limit has new degenerate ends, which hints at the fact that geometric limits of quasi-Fuchsian groups can be quite general.