A Survey of Schottky Groups

Talk by Jane Gilman

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Schottky Groups and Their Relatives-

Given a matrix $h \in SL_2(\mathbb{C})$, $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, h may be interpreted as the Möbius transformation

$$z \mapsto \frac{az+b}{cz+d}$$

As the matrices A and $-A$ give the same Möbius transformation, the matrix group that is actually isomorphic to the group of Möbius transformations is $PSL_2(\mathbb{C}) := SL_2(\mathbb{C})/\pm I$. The action of a Möbius transformation on the Riemann sphere extends to an action by isometries on $\mathbb{H}^3$.

If G is a discrete subgroup of $PSL_2(\mathbb{C})$, then $\mathbb{H}^3/G$ is a 3-manifold, and $\partial(\mathbb{H}^3/G)$ is a disjoint union of Riemann surfaces.

Möbius transformations are classified based on the geometric properties of their action on the hyperbolic plane as either elliptic, parabolic, or loxodromic. Recall the following connection between algebra and geometry for $[A] \in PSL_2(\mathbb{C})$:
- if $|\mathrm{tr}(A)| < 2$ then $[A]$ is elliptic
- if $|\mathrm{tr}(A)| = 2$ then $[A]$ is parabolic
- if $|\mathrm{tr}(A)| > 2$ then $[A]$ is loxodromic

The topic of this talk is Schottky groups. Classical Schottky groups are finitely generated subgroups of $PSL_2(\mathbb{C})$ generated by isometries sending the exterior of one circular disc to the interior of a second circular disc disjoint from it. There are many ways of altering this definition. One generalization is to replace the circular discs in the definition with discs whose boundaries are general Jordan curves. This gives the definition of a Schottky group. A Schottky group that is not classical Schottky on any set of generators is a non-classical Schottky group.

A group of Möbius transformations is called a *marked classical Schottky group* on the ordered set of generators $T_1, ..., T_g$, if there exist disjoint circles $C_1, C_1', C_2, C_2', ..., C_g, C_g'$

bounding a $2g$-connected domain D in the Riemann sphere such that $T_i(D) \cap D = \varnothing$ and $T_i(C_i) = C_i'$. Again, allowing the circles to be Jordan curves gives the definition of a *marked nonclassical Schottky group.*

The number of generators g is called the genus of the Schottky group. It is not clear that these altered definitions actually yield distinct classes of groups. The following theorem shows that the marked Schottky groups are no less general than unmarked Schottky groups.

THEOREM: (Chuckrow) [5,6,7] If Γ is a Schottky group of genus g, then Γ is a marked Schottky group on every set of g generators.

Whether or not non-classical Schottky groups are actually more general that classical Schottky groups is not at all obvious. Marden [14] eventually proved the existence of nonclassical Schottky groups, but his proof was nonconstructive. Yamamoto [16] gave the first explicit example of a Schottky group that is not classical Schottky.

Details will not be given in this lecture, but there is an object called Schottky space which serves as the moduli space of Schottky groups of a given genus. This space has a well defined boundary. Every group on the boundary of Schottky space is either a Kleinian group whose quotient manifold has a cusp, or a free strictly loxodromic group which fails to be discrete.

Schottky groups with tangent circles at parabolic fixed points of the marked generators are called noded Schottky groups. For noded Schottky groups tangencies are only between the paired circles C_i and $g_i(C_i) = C_i'$. However, there is reason to allow tangencies of the Schottky circles that are not necessarily paired by a generator and at points that are not necessarily parabolic fixed points. Such groups are called T-Schottky groups, where T means *any* tangency allowed.

T-SCHOTTKY GROUPS WITH TWO PARABOLIC GENERATORS-

We will consider in more depth the case of a rank 2 T-Schottky group, generated by two parabolic elements. We can normalize in such a way that our group admits a presentation of the form

$$G_\lambda = \langle S, T_\lambda \rangle, \text{ where } S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \; T_\lambda = \begin{pmatrix} 1 & 2\lambda \\ 0 & 1 \end{pmatrix} \text{ for } \lambda \in \mathbb{C}, \; \lambda \neq 0$$

A natural question is which values of λ make G_λ a free group. Lyndon and Ulman [13] discovered regions in the λ plane where G_λ is a free group for λ in these regions. These regions are symmetric about the real and imaginary axis. Portions of three of the regions in the first quadrant are shown on the first page of figures.

Thinking about these groups in terms of their quotient Riemann surfaces, another natural question to ask which values of λ yield surfaces of a given topological type. For G_λ, there are two possible topological types: the four-times punctured sphere with two pairs of punctures identified, and a pair of three-times punctured spheres. The second page of figures shows the Riley slice, the interior of which is the locus of λ such that G_λ's quotient surface is a four-times punctured sphere. The groups that yield four-times punctured sphere quotients admit deformations, so this explains why this locus is open. Groups yielding pairs of three-times punctured sphere's are called "cusps."

Keen and Series [12] studied the boundary of the Riley slice by looking at a certain countable collection of polynomials. These polynomials were given by the trace of the elements representing a simple closed on the quotient surface of slope $\frac{p}{q}$. Looking at the set of λ where this trace is real gives a collection of lines from infinity to the boundary of the Riley slice. The intersection point of this ray with the Riley slice is dentoted $\lambda_{\frac{p}{q}}$ (see [10]).

One tool for studying the set of λ where G_λ is a free group is Jørgensen's inequality, which says that for a non-elementary Kleinian group $\langle S, T\rangle$, $|\text{tr}[S,T]-2|+|\text{tr}^2 S-4| < 1$. Applying this to the context above, it gives that if $|\lambda| < \frac{1}{2}$ then G_λ is not discrete.

Theorem: (Gilman, Waterman) The boundary of classical T-Schottky space for the family of groups G_λ is given by $\{\lambda = x+iy \ : \ |y| = 1-\frac{x^2}{4}\}$.

There is a standard method of analyzing groups going back to Jordan, which is to find a sequence of subgroups defined by an iterative process. The sequence of iterated subgroups is listed below. As seen in [10], each iterated subgroup can be renormalized and is conjugate to the renormalized G_λ given below where "$\sim$" is denotes isomorphic.

$G_\lambda \sim \langle S, T\rangle$
$G_{\widetilde{\lambda}} \sim \langle S, TST^{-1}\rangle$
$G_{\widetilde{\widetilde{\lambda}}} \sim \langle S, (TST^{-1})S(TST^{-1})\rangle$
. . .

In terms of λ, $\widetilde{\lambda} = -2\lambda^2$, so in order to understand the effect of iterating subgroups in this manner we need only to understand the effect of iterating the function $f(z) = -2z^2$. We call a group *n-th classical T-Schottky* if $G^n_f(\lambda)$ is classical T-Schottky and $G^{n-1}_f(\lambda)$ is not. Using the Gilman-Waterman boundary and iteration ([10]) it can be shown that this definition makes sense. The boundaries of n-th classical T-Schottky space are simply the iterates of the boundaries of T-Schottky space. Taking this information into consideration we can see a lot more structure in the parameter space. A structure theorem for two parabolic space was obtained by Gilman [10]. A figure showing some of the finer structure of the parameter space obtained by Gilman is given on the third page of figures.

Corollary: (Gilman) If $f^n(\lambda) = \lambda_{\frac{p}{q}}$ then either G_λ is not free or G_λ has additional conjugacy classes of primitive parabolics, in which case the group is called a parabolic dust group.

An open problem is to calculate the values of $|\lambda_{\frac{p}{q}}|$. Another open question, asked by Ian Agol, (see [10]) is where the $\lambda_{\frac{p}{q}}$ lie in the tesselation given by mapping a circle of radius one about the origin under f. It is also not known whether this discussion can be replicated for arbitrary representations into $PSL_2(\mathbb{C})$.

Summary of Higher Dimensional Results-

For a discrete group Γ, let $\Lambda(\Gamma)$ denote the limit set. We conjecture that the Hausdorff dimension of $\Lambda(\Gamma)$ is a function of the infimum of the distance between the centers of the circles, and the radii of the circles. The following results relate to this conjecture.

Let Γ be a discrete group of hyperbolic isometries, and ζ a hyperbolic metric. The Poincaré series associated to Γ is $\sum_{\gamma\in\Gamma} e^{-s\zeta(0,\gamma(0))}$. There exists an s_0 such that for all $s > s_0$, the series converges. This number s_0 is denoted $\delta(\Gamma)$ and is referred to as the Poincaré exponent of Γ. The following theorem relates the Poincaré series to the limit set.

Theorem: (Bishop-Jones) If Γ is a non-elementary discrete group of hyperbolic isometries then $\delta(\Gamma)$ is equal to the Hausdorff dimension of the limit set.

Let $\lambda_0(\mathbb{H}^3/\Gamma)$ denote the lowest value of the Laplacian on $\mathbb{H}^3/\Gamma$.

Theorem: (Doyle) For any classical Schottky group Γ with two generators, there exist universal constants L_2 and U_2 such that $\lambda_0 > L_2 > 0$ and $\delta(\Gamma) \leq U_2 < 2$.

As a corollary, we have that if $\delta > U_2$, then Γ is non-classical Schottky. Gilman and Malek are using this result to give a new proof that Yamamoto's example is not a classical Schottky group. While it is known that universal constants L_2 and U_2 exist, what the exact values of these constants are stands as an open question.

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$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad T_\lambda = \begin{pmatrix} 1 & 2\lambda \\ 0 & 1 \end{pmatrix}$$

$$\lambda \in \mathbb{C}, \quad \lambda \neq 0$$

$$G_\lambda = \langle S, T_\lambda \rangle$$

FOR WHAT VALUES of λ is G_λ a free group?

Lyndon-Ullman 1969

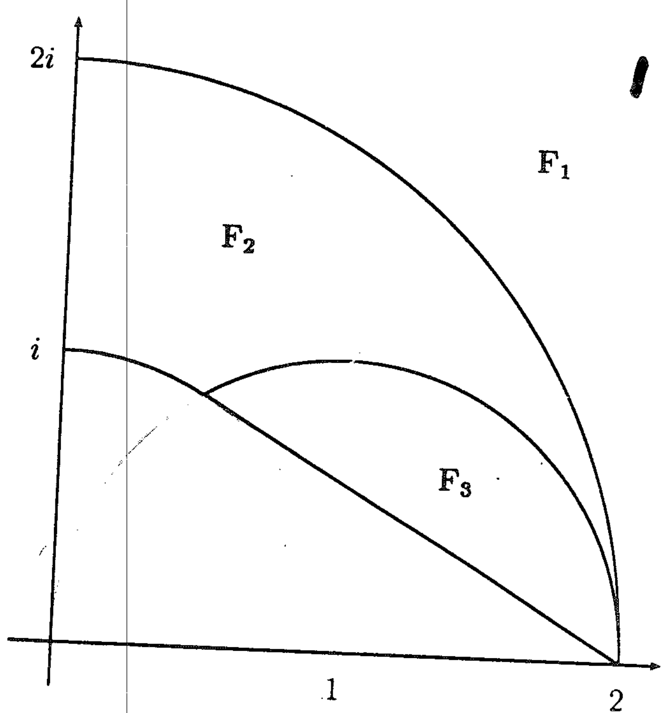

Figure 8: Lyndon - Ullman Domain

1st quadrant $\mathbb{C}$

λ plane

1/a

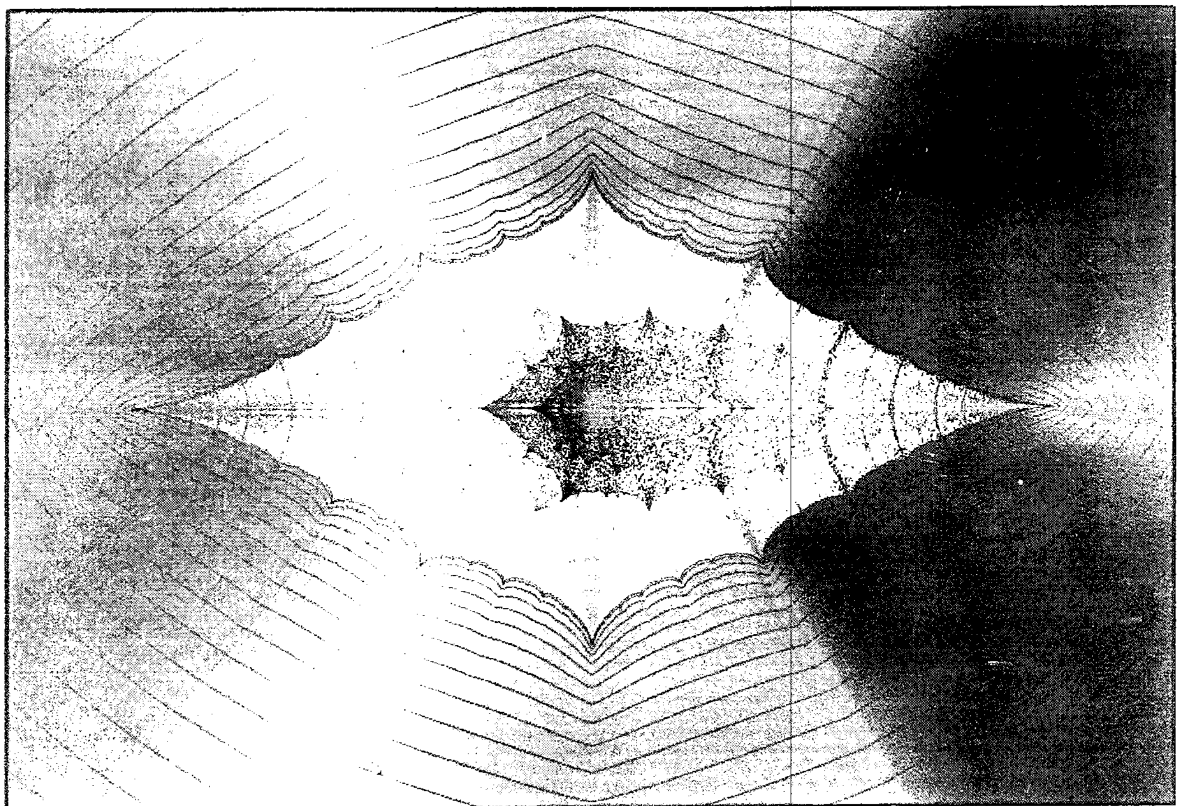

Figure 1: The Riley Slice: Courtesy of David Wright ~1980

Discrete-Free Quot ≅ 4 punctured sphere

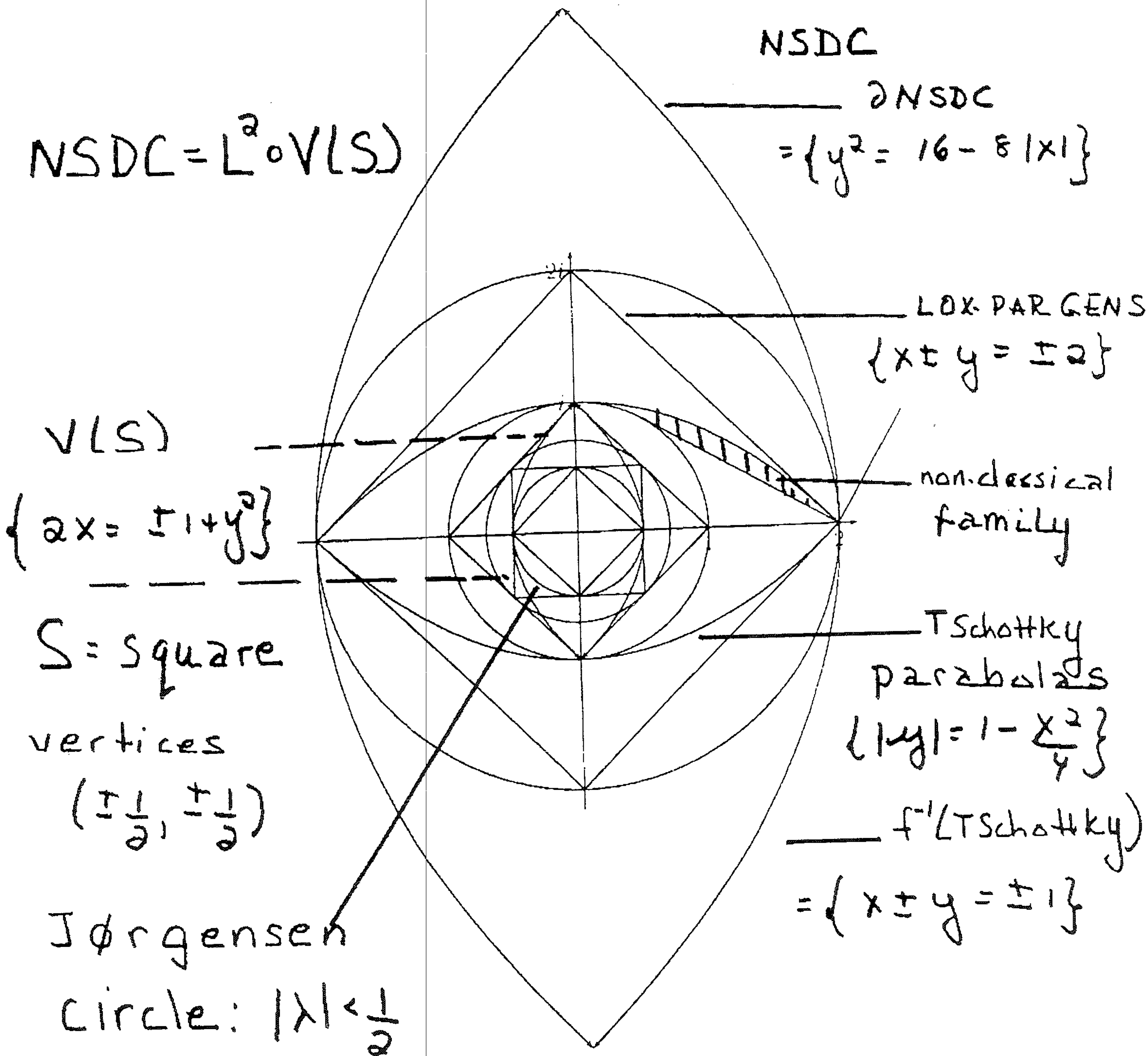

FIGURE 1. **Boundary Regions** Each point $\lambda \in \mathbb{C}$ corresponds to a two-generator group. The shaded region shows $\mathcal{NCF}$ the one parameter family of non-classical T-Schottky groups in the first quadrant. The exterior to the outer parabolas are the non-separating disjoint circle groups (NSDC groups), the middle parabolas are the boundary of the classical groups. The boundary of the Riley slice includes ± 2 and $\pm i$ but otherwsie lies interior to the Schottky parabolas. Points inside the Jorgensen circle ($|\lambda| < \frac{1}{2}$) are non-discrete groups. Between the Riley slice and the Jørgensen circle are additional non-classical groups together with degenerate groups, isolated

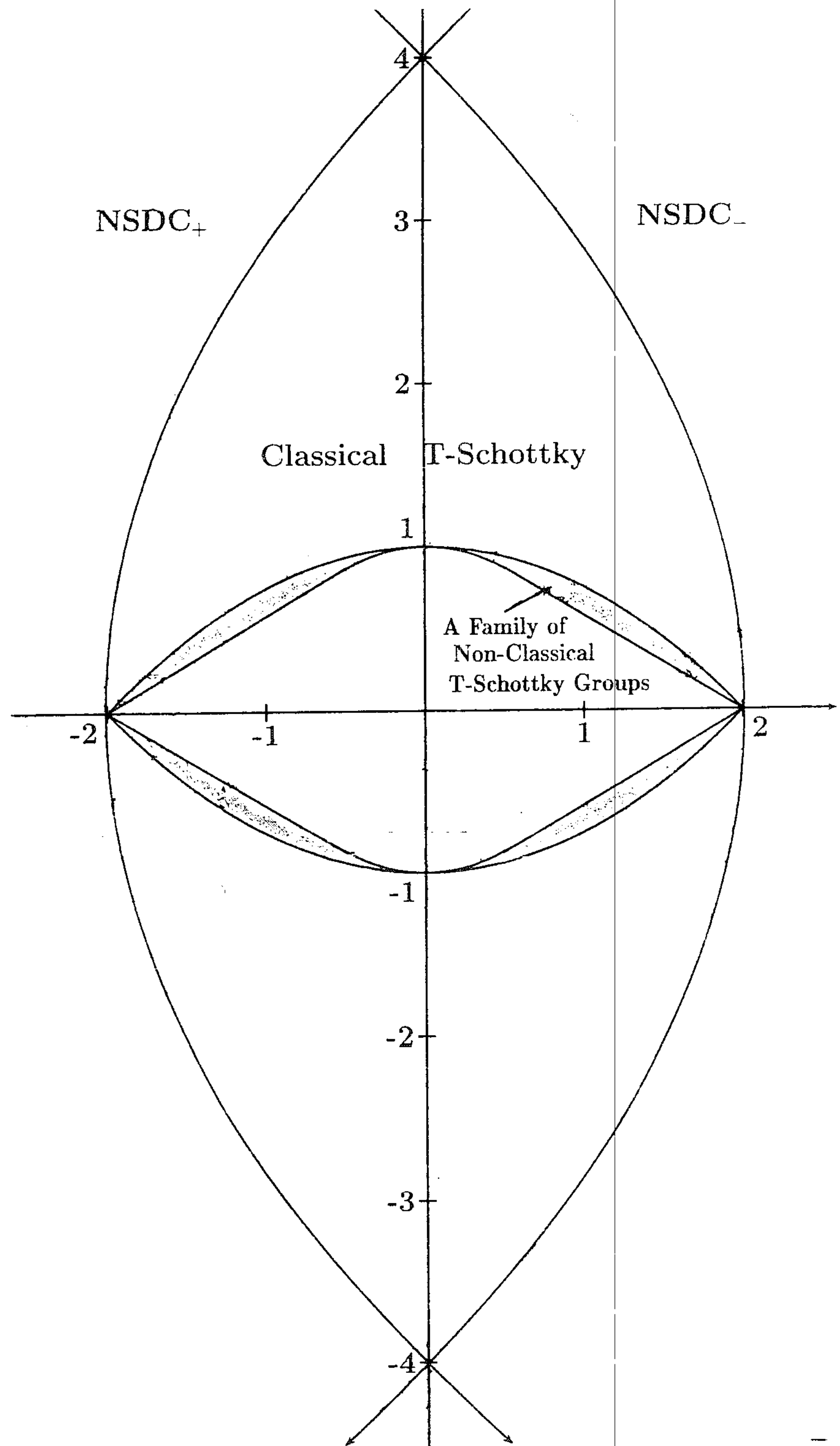

Figure 7: **Superimposed Boundary Parabolas.** Each point $\lambda \in \hat{\mathbb{C}}$ corresponds to a two generator group. The darkest region shows a one parameter family of non-classical T-Schottky groups. The line-shaded subset of the classical T-Schottky groups comprises the non-separating disjoint circle groups (NSDC groups). The unshaded region consists of additional non-classical T-Schottky groups together with degenerate groups, isolated discrete groups and non-discrete groups. Points inside the Jørgensen circle ($|\lambda| < \frac{1}{2}$) are non-discrete groups.