Last Times:
- Defined CVn - Either as Space of Graphs or as Space of Trees
- Defined 3 Topologies that Coincide

Thm: CVn is Contractible (Culler-Vogtmann)

[Will then know that Out(Fn) acts properly on CVn w/ finite stabilizers, thus giving cohomological information about CVn]

(This is Skora's Proof)

Prf: In order to show that a space is contractible, it suffices to show that we have a basepoint and then that every point can be joined to this basepoint by a path in a canonical way.

Choose Basepoint in CVn:

- Basepoint will be a Cayley tree To of Fn (equivalent to choosing some basis for Fn)
  But the metric on To will actually depend on the tree we are trying to join to To
    - We are thus actually considering as a basepoint the simplex containing To
      [obtained by varying the lengths of the edges in To]
    - This differentiation between trees/simplices as basepoints is inconsequential since simplices are contractible

- Consider any tree Te CVn w/ an action of Fn which is free, isometric, & minimal
  - Eventually we will want to find a more or less canonical path from T to To
  - For now, we want to choose a metric on To that depends on T

Metric:
- Choose equivariant f: To \to T [f(gx) = gf(x) V ge Fn]
  - f(1) = some point Pe T [have a choice, but for now choose any point P]
  - Since f is equivariant, we know where every vertex goes, i.e. f(gr) = gr P
  - Extend this map linearly to the edges

\[ \begin{array}{c}
  T_0 \\
  \text{f} \quad \text{T} \\
\end{array} \]
Change Metric on $T_0$:
- For each of $n$ loops have a length
  - In our $F_n$ case, have exactly 2 parameters: If think of edge as being associated to either generator $a$ or $b$, have a choice of lengths for these 2 edges.
  - In quotient space of $F_n$ action, have $n$ loops that we can choose length for.
- Change metric so that the restriction of $f$ to any edge of $T_0$ is an isometry.
  - Edge associated to generator $a_i$ in $F_n$ has length $d(f, P, a_i P)$

Remark: Keep in mind that this metric depends on $T$, which is why we're actually considering a simplex for our basepoint $T_0$.

Construct Path between $T_0$ and $T_0$:
- $f: T_0 \rightarrow T$ is an equivariant morphism
  - i.e. Each segment (between 2 pts) in $T_0$ may be subdivided into finitely many subsegments so $f|_{\text{subsegment}}$ is an isometry.
  - [As we have done the construction, any edge may be taken as a segment.]
- Given an equivariant morphism $f: T_0 \rightarrow T=T_\infty$, we can continuously deform from $T_0$ to $T$.
- Construct intermediary trees $T_t$, $0 \leq t \leq \infty$; morphism $T_0 \rightarrow T_\infty$ factors through $T_t$.
  - Understood that, as $t \rightarrow \infty$, $T_t \rightarrow T_\infty$.

Remarks:
1. In the case of $CV^n$, $f$ to $st\ T_\infty = T_\infty$ for $t \geq 0$ (not necessarily true in general)
2. Remember that for $T \neq T'$, $T_0$ will be combinatorially same as $T_\infty$ (possibly different edge lengths)
3. The main property of the deformation that we will need to discuss is continuity.

Need continuity to know that these paths are close:

2. Constructions of $T_t$:
   1. Intuitive Idea:
      - Have equivariant morphism $T_0 \rightarrow T_\infty$.
      - If morphism is an isometry, don't have to do anything.
      - If morphism is not an isometry globally, then folding occurs.
Folding: *Have vertex i 2 edges
*Remember that edges don't have to have same length because have changed lengths in T
*Initial segments of the 2 edges are Folded

Whenever edges are folded, for t small, get T_t by folding the edges along a segment of length t

Let t grow until something happens (qualitative change):
- Edges of 2 different lengths are folded Cannot fold anymore
- Combinatorics of the tree is changed
- Some vertex is folded

Find 1st value of t for which this happens
- Then stop, go get coffee, start again (Look at what edges are Folded? continue)

Problem w/ This Defn:
May have to drink more & more coffee
i.e. May happen that accidents accumulate & then we'll be in trouble
They don't, but it's not clear at this point

2) Correct Definition of T_t:
- Have f: T_o → T
- Can always factor through T_t: T_o → T_t → T
  [Maps are onto since they're morphisms: morphisms between minimal trees are always onto]

Going to define T_t as a space & view T_t as a quotient of T
So need to understand when 2 points in T have the same image in T_t
When are $x, y \in T_0$ identified in $T_\pm$?

- Necessary condition is that they should be identified in $T_{oo}$
  
  [If have distinct images in $T_{oo}$, definitely have distinct images in $T_\pm$]

- So suppose $f(x) = f(y)$; look at the segment in $T_0$ between $x$ and $y$

  \[ T_0 \xrightarrow{\sim} T_\pm \]

- Look at the image in $T_{oo}$

  \[ T_0 \xrightarrow{f} T_{oo} \]

  (This is the image of the segment)

  \[ f(x) = f(y) \]

  (Have assumed that $x$, $y$ have same image)

- Remember that we've assumed that this map is isotropic, so we can subdivide the segment in $T_\pm$ into finitely many subintervals so that the map is an isometry on each subinterval

- Look at the maximal distance from $f(x)$ & compare this with:

  \[ x, y \text{ identified in } T_\pm \iff \text{Maximal Distance is } \leq \pm \]

  (Image of segment is subtree; look at how far away this subtree goes from $f(x) = f(y)$)

This defines $T_\pm$ as a set, also need to define a metric, which only describe briefly:

- From $T_0 \rightarrow T_{oo}$ factoring through $T_\pm$, have map $T_0 \rightarrow T_\pm$

- Metric on $T_\pm$ is the maximal metric w/ the property that $T_0 \rightarrow T_\pm$ is a $1$-Lipschitz map (doesn't increase distance)

Remarks:

1. $T_0 \rightarrow T_{oo}$ is an isometry (of simplicial trees w/ free action of $F_n$) tells us that if $x, y$ have same image in $T_{oo}$, then $\exists$ a uniform bound for their distance $T_{oo}$, when $\pm >$ this uniform bound, $T_\pm = T_{oo}$

   (This is the comment earlier that, in $C(V, I \rightarrow 0)$, $T_\pm = T_{oo}$ $\forall \pm > t_0$)

2. It is obvious that the maps $T_0 \rightarrow T_{oo}$ factor through $T_\pm$
Continuity:

\[ T_0 \xrightarrow{f} T \]

Want that the deformation between these paths is continuous

Issues w/ Continuity:

1. In order to define a path \( T_0 \xrightarrow{f} T \), we need a morphism \( f : T_0 \to T \).
   - We have to make a choice for that; we need that \( f \) depends continuously on \( T \).
   - [Continuity wrt Gromov-Hausdorff Equivariant Topology or the space of morphisms]

2. \( T_t \) depends continuously on \( f \times t \).
   - (Once you have \( f \), you can define this intermediate tree \( T_t \); this should also be continuous)

A Few Words About \( f \):

- \( f \) was constructed using basepoint \( P \in T \).
  - This was the only choice that we had to make because, once we had the basepoint \( P \), the image of \( P \), other points were mapped equivariantly by extension on the edges.

- Need that \( P \) depends continuously on \( T \).
  - i.e., for any \( g \in F_n \), \( d_T(P_t, gP_t) \) depends continuously on \( T \).
  - (It makes sense to talk about continuity since this is a function of \( T \) only, i.e., once you fix \( g \) this is a function that only depends on the tree \( T \)).

- This tells us that the length assignment on \( T \) changes continuously.
  - Length assignment of \( T_t \) varies continuously for each edge.
  - (Distance \( d_T(P_t, gP_t) \) associates length to a generator, representing an edge).

- So we need to be able to find a basepoint in a clever way to mean that this will be continuous.

Skora gave a construction based on minimization:

- Look at generators of \( F_n \); a point \( t \) how much the generators move this point.
  - Take the max over the generating set; you have to minimize that.

We're going to use a different construction:

- Fix \( h_1, h_2 \) hyperbolic in \( F_n \).
- In your tree \( T \), they'll have axes like last time.
3 Possibilities for the Axes of the 2 elements $h_1, h_2 \in F_n$:

- Choose basepoint as the point of $A_{h_1}$ on the bridge between $A_{h_1}$ & $A_{h_2}$
- Take intersection as basepoint
- Point as above, relying on orientation of $A_{h_1}$

- You can show that this is a continuous choice of basepoints
  (Point is that, by looking at these 2 elements, can select some region in the tree, get your basepoint from there)

(2) is Technical; a Pain: Use G-H Topology, Lecturer wrote an account, as did Max Clay

Question from Audience: What if don't get a minimal tree?
  - Just restrict to minimal subtree (what's not in minimal subtree)
  - Maybe in case of $CV_n, T_e$ is always minimal, doesn't remember, but it's not very important

Next Topic: Train Tracks [Introduced by Bestvina-Handel] Later developed by Bestvina-Feigh-Handel

Notation: $\Phi \in \text{Out}(F_n), \phi \in \text{Aut}(F_n)$

Special Cases:
- $\Phi$ is Geometric, if it is induced by a homeomorphism of a compact surface $\Sigma$ with $\pi_1(\Sigma) \cong F_n$
  [Automorphism is induced by a surface homeomorphism]
  - If you have a compact surface, its fundamental group is free
  - Whenever you have a homeomorphism, you know what to do because you can use Nielsen-Thurston Theory to analyze it

In Particular, If $n=2$:
- Any automorphism of $F_2$ is geometric (Known due to Nielsen)
  i.e. Every outer-automorphism $\phi$ of $F_2$ comes from a homeomorphism of the punctured torus
After raising $\Phi$ to a power, 3 possibilities:

1) Identity (if finite order)

2) Power of a Dehn Twist (This is what reducible means in this case)

   Example: $a \rightarrow a$
   $b \rightarrow ba$

   [Lecturer thinks this is the one that we played with on Monday in our "Little Hydra Game"]

3) Pseudo-Anosov

   Example: $a \rightarrow ab$
   $b \rightarrow a$

   [See Tim Riley’s talk on Hercules! the Hydra]

Play the same game as in Professor Riley's talk:

- Start with word $w$
- Remove 1st letters; then let other letters regenerate according to this rule
- Question is whether word completely disappears
  - in which case say that Hercules wins
  - word keeps expanding
    - Hercules is in big trouble

In Pseudo-Anosov case, Hercules is a loser

We will give an argument showing that Hercules loses against any positive word $w$ of length $\geq 5$

- $a \rightarrow ab \rightarrow aba$

- $b \rightarrow a \rightarrow ab$  $\Rightarrow$ After applying algorithm twice, length is at least doubled

Consider $|w| \geq 5$:

- If Hercules strikes twice, get $w_1$ of length $\geq 6$
  - $[1 \times 2 \text{ letters disappear, but then have } \geq 3 \text{ letters remaining, then giving } \geq 6 \text{ letters}]
- If Hercules strikes again, get $w_2$ of length $\geq 8$
  - $[2(6-2) = 8]$
- If Hercules strikes yet again, get $w_3$ of length $\geq 12$
  - $[2(8-2) = 12]$

  In this case, the Hydra grows exponentially, so Hercules doesn’t win

Remarks: Gave this argument because it shows what happens here

- Have a word $w$, it expands exponentially (everything is multiplied by at least 2)
- Then something happens near the edges, which you don’t quite control
- Point is that exponential growth is strong enough that, even with some damage done near the ends, exponential growth is going to prevail

This is an argument that we’ll see later when we do train tracks (legal paths on train tracks)
We talked about geometric automorphisms, which come from homeomorphisms of surfaces, which can be analyzed via Thurston-Nielsen Theory.

For $n=2$, all automorphisms come from surfaces, this is not true in $n=3$.

**Example:** For $n=3$, Automorphisms not always geometric

- Let $a, b, c$ generate $F_3$.
- $\alpha: \begin{array}{ccc} a & \rightarrow & a \\ b & \rightarrow & ba \\ c & \rightarrow & cb \end{array}$ is not geometric.

Look at what happens when start with $c$ and iterate the automorphism on $c$:

- $c \rightarrow cb \rightarrow cbba \rightarrow cbbaba \rightarrow cbbaba^2 \rightarrow \cdots$
- Length of $\alpha^p(c) = \frac{p^2 + p + 2}{2}$

What's important is that it's quadratic in $p$.

- This word is cyclically reduced, so, more generally:
  - Length of $\phi^p([c])$ was quadratic

(If word not cyclically reduced, reduce it (in such a case, the lengths may decrease).

**Point:** Because of this, the automorphism cannot be geometric.

- If $\phi$ geometric, the growth of $\phi^p([c])$ is linear or exponential (cannot be quadratic).
- After taking a power, the homeomorphism is made of Dehn Twists! Pseudo-Anosov.

(Grows Linearly, Grows Exponentially.)

(If there's a curve passing through region where the homeomorphism is pseudo-Anosov, length of shortest curve in conjugacy class will grow exponentially. If only passes through regions with Dehn twists, it will grow linearly. But it can never grow quadratically.)

So, whenever we see quadratic growth, such as in this case, we know that the homeomorphism is not geometric.
(2) (2\textsuperscript{nd} Special Case) Automorphism $\Phi$ is induced by a homeomorphism (automorphism) of a graph

- Symm(1) has 12 elements
  - Can permute the 3 edges: This gives 6 automorphisms
  - Can flip (symmetry over horizontal line)
- This defines a subgroup of order 12 in $\text{Out}(F_n)$
- This is a construction that gives automorphisms of finite order in $\text{Out}(F_n)$
- There is a converse to this construction:

**Theorem (Zimmerman–Culler–Kharitonov):**

Any finite $F \preceq \text{Out}(F_n)$ comes from symmetries of a graph

- Proof is based on structure of virtually free groups
- This has a corollary: $F$ has a fixed point in $CV_n$

**Consequence:** Since the action $\text{Out}(F_n) \curvearrowright CV_n$ has finitely many orbits of simplices, up to conjugacy, there are only finitely many finite subgroups in $\text{Out}(F_n)$

"Now that we've exhausted the special cases... We have to really do something."

**Take any $\Phi \in \text{Out}(F_n)$, a positive homomorphism**

For example, $a \mapsto aba$

$b \mapsto ba$

What's nice about positive automorphisms is that:

If you start with a generator; apply the automorphism over and over, you are never going to see any cancellation, i.e. No cancellation in computing $\Phi^p(a), \Phi^p(b)$

This is the property that we'd like to have in general

Note: If you start with another word, such as a word that's not positive, other things may happen. For instance, in this example:

$aba^{-1}b^{-1} \rightarrow aba^{-1}b^{-1} a^{-1} b^{-1} a^{-1} b^{-1} = aba^{-1}b^{-1}$

Cancellations
The property that, when you start with the generators, you don't have any cancellation is the property that we can hope would hold in general. This leads us to the idea of train tracks.

-Sometimes an automorphism is positive & sometimes not

Example: \( a \rightarrow c \) is not a positive automorphism
\( b \rightarrow c^{-1}a \) & not nice is the sense above because we have the cancellation: \( c \rightarrow c^{-1}b \rightarrow b^{-1}c^{-1}a = b^{-1}a \)

We will try to get rid of the cancellation \( cc^{-1} \):

-What does this say geometrically?
-Whenever I have an automorphism like this, I can realize it as a map on the nose:

The vertex is mapped to the vertex \( a \):

Problem in this case is that images of \( b \) & \( c \) both start with \( c^{-1} \):

\[
\begin{align*}
    a & \rightarrow c \\
    b & \rightarrow c^{-1}a \\
    c & \rightarrow c^{-1}b
\end{align*}
\]

- Stretch the loop over \( c \) in the negative direction then over \( a \)

From this data, I have a map on the graph realizing the automorphism:

- We have an illegal turn because, when we apply the map, \( b \) gets folded.

- That in itself is not a problem. In any positive automorphism like this, there will be illegal turns.
- What is a problem is that the image of \( c \) goes over an illegal turn.
- This is why we have cancellation (When apply map to \( c^{-1}b \) get cancellation)

We will solve this problem by changing the graph:

- We must subdivide \( b \) & \( c \) on the left because they are sub-divided on the right.
- Give new names to our new edges.

\[\text{Graph After Folding}\]

\[\text{[This is a graph with 2 vertices but we're used to the fact that maybe sometimes we need to have graphs with more than 1 vertex]}\]
Can define a new map on the graph on the right (graph after folding):

A → DC  [image of a was e, which is now called DC]
B → a  [B is 2nd half of b: 1st half went to c⁻¹]
C → DB  [C is 2nd half of c, which went to b, b is now called DB]
D → C⁻¹D⁻¹  [D is 1st half of b, which went to c⁻¹, which is now called C⁻¹D⁻¹]

This was the 1st step of changing the map; we had to change it to get a nicer map
But are things actually better?
Not really, they even look worse.

For instance, DC is now an illegal turn since it's in the image of a & D
[If you look at the inverse of DC, there's cancellation]

So we will do one more folding

What does the map look like now?

a → D̂C
B → a
C → B
D → C⁻¹ = C⁻¹e⁻¹
E → D = D̂e⁻¹

Now collapse e, i.e. Collapse e to the left

a → D̂C
B → D̂⁻¹a  [New image of B' = Old Image of e⁻¹B = e D̂⁻¹a]
C → B'
D → C⁻¹
Rewrite w/ Better Notation:

\[ \begin{align*}
  a & \rightarrow dc \\
  b & \rightarrow d^{-1}a \\
  c & \rightarrow b \\
  d & \rightarrow c^{-1}
\end{align*} \]

This new map represents the same automorphism, but doesn't have any cancellation.

This is a **Train Track Map**.

**Important Property:** There is no cancellation when iterating the map on an edge.

(Almost as good as one of the positive maps talked about above)

In 3 Weeks (Next Lecture): Train Tracks & Relative Train Tracks