Note Taker Checklist Form - MSRI

Name: Ioana Mihailea

E-mail Address/ Phone #: imihaila@csupomona.edu

Talk Title and Workshop assigned to:
Geometric Group Theory by Example

Lecturer (Full name): James Cannon
Date & Time of Event: 8/27/07 9-10 am

Check List:

( ) Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.

( ) Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.

( ) Take down all notes from media provided (blackboard, overhead, etc.)

( ) Gather all other lecture materials (i.e. Handouts, etc.)

( ) Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do NOT use pencil or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords: curvature, geometry, geometric action, isometry, cocompact, properly discontinuous, Cayley graph, word problem, recognition theorem, Todd-Coxeter algorithm, Bridson

2. Please summarize the lecture in 5 or less sentences.

   Through various examples, lecturer conveyed the following ideas: finitely generated groups act on geometries, the growth of a groups in planar schwarzian, and characterized by (infrernal) derivation. The most typical behavior is negative curvatures, no fixed point, no minimum or maximum curvature. The materials on the list above are gathered, please scan all materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

For Video Tapings-MSRI 9/2006
Nonpositive & negative curvature in group theory.

Sectional) curvature is a measure of how fast a space is expanding in its various planar directions.

Group applications require attaching (associating) spaces to groups.

Curvature has profound effect on structure & algorithmic properties.
NEGATIVE & NONPOSITIVE CURVATURE IN GROUPS

GEOMETRIC GROUP THEORY—
THE STUDY OF GEOMETRIC GROUP ACTIONS

\[
G \times X \xrightarrow{\text{action}} X
\]

1 \cdot x = x

\((g \cdot h) \cdot x = g \cdot (h \cdot x)\)

EXAMPLE

\[G = \mathbb{F}_2 = \langle x, y \rangle\]

\[X_G = \ldots \ldots \ldots \ldots = \text{TREE, REGULAR, 4-VALENT.}\]

APPLICATION. EVERY SUBGROUP OF A FREE GROUP IS FREE.
(TOOLS: \(\mathbb{T}_1\) AND COVERING SPACES.)
EXAMPLE.

\[ H = K_2 = \langle a, b \mid a^2 = b^2 \rangle \]

\[ = \langle a, b \rangle / \langle \langle a^2 b^2 \rangle \rangle \]

RELATION

RELATOR

NORMAL CLOSURE

APPLICATION: \( F_2 \not\rightarrow K_2 \).

PROOF: \( X_H \) (CURVATURE = 0) CANNOT HOLD \( X_G \) (CURVATURE < 0).
1.3. GEOMETRIC ACTIONS

A GEOMETRY IS A SPACE $X$ ENDOWED WITH A PROPER PATH METRIC $d$.

Closed Balls $d(x,y)$ is realized by a path

The action $G \times X \to X$ is geometric if it is

1) ISOMETRIC: $d(gx,gy) = d(x,y)$
2) COCOMPACT: $X/G$ is compact
3) PROPERLY DISCONTINUOUS: $\forall K$ compact
   $\{ g \in G \mid K \cap gK \neq \emptyset \} \text{ is finite.}$

Theorem. If $G \times X \to X$ is geometric, then the rough global structure of $X$ is determined by $G$. 
1.4

The Cayley Graph

Theorem. If $G$ is finitely generated by generating set $C = C^1$, then $G$ acts geometrically on its Cayley graph $\Gamma = T(G, C)$.

\[
\Gamma = (V, C, E)
\]

\[V = G\]
\[C = C\]
\[E = \{ e = (v, c, v.c) \mid v \in V, c \in C \}\]
\[e^{-1} = (v.c, c^{-1}, v)\]

$e$ and $e^{-1}$ define the same undirected edge

\[G \times \Gamma \to \Gamma\]
\[(g, v) \mapsto g.v\]
\[g(v, c, v.c) = (gv, c, g.v.c)\]
1.5 The Word Problem: Constructing $\Gamma$.

Example: $\langle x, y \mid x^2 = y^5 = (xy)^3 = 1 \rangle$

Relator pictures:

$\Gamma$:

Recognition theorem: $\Gamma = \Gamma(G, C)$ iff

$RT(1)$ Each color enters and exits each vertex exactly once.

$RT(2)$ Each relator labels a closed path at each vertex.

$RT(3)$ Each path closure is forced by (1) and (2).
**Todd-Coxeter Algorithm**

**Step 0:** Set $T_0 = \{ \cdot \}$
(A single vertex)
Assume $T_{n-1}$ Defined.

**Step (1)$_n$:** Add labelled edges to $T_{n-1}$ so as to satisfy RT(1)

**Step (2)$_n$:** If any relator labels a path that is not closed, identify the endpoints of that path.

**Step (3)$_n$:** If two edges with the same label exit the same vertex, identify those edges (repeat as necessary).

Repeat (1)$_n$, (2)$_n$, (3)$_n$ until the graph no longer changes.
\[ F_2 \rightarrow K_2 \ (\text{Details}) \]

For \( N > 0 \), \( K_2 \) has exactly \( 4N \) vertices at distance \( N \) from the origin, or

\[
1 + 4(1+\ldots+N) = 1 + 4 \cdot \frac{N(N+1)}{2}
\]

vertices in ball \( B_{K_2}(1, N) \).

For \( N > 0 \), \( F_2 \) has exactly \( 4 \cdot 3^{N-1} \) vertices at distance \( N \) from the origin, or

\[
1 + 4(1+3+\ldots+3^{N-1}) = 1 + 4 \cdot \frac{3^N - 1}{2}
\]
in \( B_{F_2}(1, N) \).

If \( F_2 \hookrightarrow K_2 \) with \( f(B_{F_2}(1,1)) = B_{K_2}(1, K) \), then \( f(B_{F_2}(1, N)) \subset B_{K_2}(1, K \cdot N) \) impossible.
CURVATURE

CLASSICAL: (GAUSS)

$K(p) = \det \, DN(p)$

$= \text{STRETCH FACTOR}$

$= \text{PRODUCT OF EIGENVALUES}$

$= \text{PRODUCT OF EIGENVALUES}$

$= \text{OF ORIENTED AREA}$

$= \text{OF } DN(p)$

GAUSS'S GREAT THEOREM:

$K$ DEPENDS ONLY ON $(S,d)$.

$K$ GIVES A 3rd ORDER CORRECTION TO LINEAR DIVERGENCE OF GEODESICS IN $S$.

GEO $\uparrow$ GEO $\uparrow$

$K > 0$ $K < 0$

CONVERGENCE $\ln$ LINEAR DIV $\exp$, DIV
**CONSTANT CURVATURE MODELS**

\[ K = 1 \quad (\text{or} \quad S^n) \]

\[ K = 0 \quad (\text{or} \quad \mathbb{R}^n) \]

\[ K = -1 \quad (\text{or} \quad H^n) \quad (\text{Hyperbolic Space}) \]

**DISCRETE APPROXIMATION**

**TO** \( H^2 \quad \{ (x, y) \in \mathbb{R}^2 \mid y > 0 \} \)
CURIOUS PROPERTIES OF $H^2$ (OR $H^n$).

(1) VOLUME & CIRCUMFERENCE ARE EXPONENTIAL FUNCTIONS OF RADIUS.

(2) EQUIDISTANT CURVES ARE EXPONENTIALLY LONGER THAN GEODESICS.

(3) THERE IS A NATURAL SPACE AT INFINITY.

(4) TRIANGLES ARE UNIFORMLY THIN AND HAVE ANGLE SUM $< \pi$. 
CURVATURE IS SECTIONAL CURVATURE

DIVERGING GEODESICS DEFINE A PLANE

DIFFERENT PLANES CAN HAVE DIFFERENT CURVATURES AT THE SAME POINT

$S^2 \times \mathbb{R}$

$H^2 \times \mathbb{R}$

$H - K = 1$
$V - K = 0$

$H - K = -1$
$V - K = 0$
Eight geometries are especially important for 3-manifolds

Constant Curvature

$R^3$, $H^3$, $S^3$

$k = 0$  $k = -1$  $k = 1$

Product Geometries

$S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$

Twisted Geometries

$\text{Sol}$, $\text{Nil}$, $\text{PSL}(2, \mathbb{R})$

The most important?

$H^3$
The surface groups (dimension 2)

\[ G = \langle x_1, y_1, \ldots, x_n, y_n \mid \prod_i [x_i, y_i] = 1 \rangle \]

Act on the geometries

\[ \begin{array}{ccc}
K=1 & K=0 & K<0 \\
\end{array} \]

This geometric behavior is typical of many group sequences.

A few exceptional groups - then \( K<0 \),
Two interesting group sequences:

**The Fibonacci groups**

\[ \langle x_1, \ldots, x_n \mid x_{i+2} = x_i \cdot x_{i+1} \rangle \]

(indices mod n)

\[ F_5 = \mathbb{Z}_1 \]

\[ F_6 = \text{Euclidean} \]

\[ F_{2n}, \ n \text{ large} \]

\[ |F_n| = \infty, \ n \geq 9 \]

**The fake Fibonacci gps**

\[ \tilde{F}_n = \langle x_1, \ldots, x_n \mid x_{i+2} = x_i^{-1} x_{i+1} \rangle \]

Properties?
Each $\tilde{F}_n$ is a 3-manifold group. Abelianization is periodic. Is $|\tilde{F}_n| = \infty$ for large $n$? Exceptional cases?
Prototypical Results in Geometric Group Theory

Procedure: Take any rough geometric property of a geometry and characterize the groups whose geometries satisfy those conditions.
STALLING'S THEORY OF ENDS.

STRUCTURE THEORY: $|\text{ends}| = \infty$

$\Rightarrow$ THE GROUP IS AN FPA OR HNN WITH AMALGAMATION OVER A FINITE GP.
GROMOV'S GROUPS
OF POLYNOMIAL
GROWTH.

IN TERMS OF A
F.G. SET C FOR G,
SUPPOSE THERE IS
A POLYNOMIAL P(X)
SUCH THAT, FOR EACH
N > 0, G HAS ≤
P(N) ELEMENTS OF MIN
WORD LENGTH ≤ N.

THEN G HAS A NILPOTENT
SUBGROUP OF FINITE
INDEX.
SUMMARY:

F.G. GROUPS $G$ ACT ON GEOMETRIES $X$.

GROWTH OF $G$ IN PLANAR SECTIONS IS GOVERNED BY (SECTIONAL) CURVATURE OF $X$.

THE MOST TYPICAL BEHAVIOR IS NEG. CURVATURE (EXPLOSIVE)

EUCLIDEAN BEHAVIOR IS $O$ CURVATURE.

MIXED CURV $\Rightarrow$ UNPREDICTABILITY
SUGGESTED EXERCISES

(1) SHOW THAT
\[ \langle x, y \mid x^2 = y^5 = (xy)^3 = 1 \rangle \]
\[ \rightarrow \text{Alt}_5 : x \rightarrow (12)(34) \]
\[ y \rightarrow (12345) \]
\[ xy \rightarrow (245) \]
IS AN ISOMORPHISM.

(2) CONSTRUCT THE CAYLEY GRAPH OF \[ \langle x_1, y_1, x_2, y_2 \mid [x_1, y_1][x_2, y_2] = 1 \rangle. \]

(3) CALCULATE THE ABELIANIZATION OF \( \text{Fib}_n \) & \( \overline{\text{Fib}}_n \).