

Note Taker Checklist Form -MSRI

Name: Ioana Mihaila

E-mail Address/ Phone #: imihaila@csupomona.edu

Talk Title and Workshop assigned to:

Quasi-isometric rigidity I
Intro to Geometric Group Theory

Lecturer (Full name): Bruce Kleiner

Date & Time of Event: 8/28/07 9-10

Check List:

- () Introduce yourself to the lecturer prior to lecture. Tell them that you will be the note taker, and that you will need to make copies of their own notes, if any.
- () Obtain all presentation materials from lecturer (i.e. Power Point files, etc). This can be done either before the lecture is to begin or after the lecture; please make arrangements with the lecturer as to when you can do this.
- () Take down all notes from media provided (blackboard, overhead, etc.)
- () Gather all other lecture materials (i.e. Handouts, etc.)
- () Scan all materials on PDF scanner in 2nd floor lab (assistance can be provided by Computing Staff) – Scan this sheet first, then materials. In the subject heading, enter the name of the speaker and date of their talk.

Please do **NOT** use **pencil** or colored pens other than black when taking notes as the scanner has a difficult time scanning pencil and other colors.

Please fill in the following after the lecture is done:

1. List 6-12 lecture keywords: Quasi-isometry, rigidity,
 Tits boundary, Symmetric space

2. Please summarize the lecture in 5 or less sentences.

lecture defined quasi-isometry groups, stated
the fundam lemmas of geometric group theory
and gave a list of rigidity problems. Discussed
recognition theorems. Talked about boundaries
of CAT(0) & CAT(-1) spaces

Once the materials on check list above are gathered, please scan ALL materials and send to the Computing Department. Return this form to Larry Patague, Head of Computing (rm 214)

Def. A (typically discontinuous) map between metric spaces $f : X \rightarrow Y$ is an (L, A) -**quasi-isometry** if for all $x_1, x_2 \in X, y \in Y$,

$$\frac{1}{L}d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq L d(x_1, x_2) + A.$$

$$\text{and} \quad d(y, f(X)) < A.$$

Two maps $f, f' : X \rightarrow Y$ are **equivalent** if their sup distance is finite:

$$\sup \{d(f(x), f'(x)) \mid x \in X\} < \infty.$$

A **quasi-inverse** of a quasi-isometry $f : X \rightarrow Y$ is a quasi-isometry $g : Y \rightarrow X$ such that the compositions $g \circ f$ and $f \circ g$ are equivalent to the identity maps.

Def. If X is a metric space, the **quasi-isometry group of X** is the collection of equivalence classes of quasi-isometries $X \rightarrow X$, with the group multiplication given by composition.

Def. Suppose G is a group, X, Y are metric spaces, and $G \curvearrowright X$ and $G \curvearrowright Y$ are actions.

A **quasi-conjugacy** is a quasi-isometry $f : X \rightarrow Y$ such that

$$\sup\{ d(f(gx), g f(x)) \mid g \in G, x \in X \} < \infty.$$

Fundamental lemma of Geometric Group Theory. Suppose G is a group, X, Y are proper geodesic metric spaces, and $G \curvearrowright X, G \curvearrowright Y$ are two discrete, cocompact, isometric actions. Then there is quasi-conjugacy $X \rightarrow Y$. In particular, X and Y are quasi-isometric.

Def. An isometric action on a metric space $G \curvearrowright X$ is **discrete** if for every $x \in X, R \in [0, \infty)$, the set

$$\{g \in G \mid d(gx, x) < R\}$$

is finite.

Cor. If G is a finitely generated group equipped with a word metric, then the action of G on itself by left translation is quasi-conjugate to any discrete, cocompact, isometric action of G on a proper geodesic metric space.

Cor. If M is a compact connected Riemannian manifold, or a finite connected polyhedron equipped with a piecewise Riemannian metric, then $\pi(M)$ is quasi-isometric to the universal cover \tilde{M} .

Rigidity problems

1. Given metric spaces X and Y , is there a quasi-isometry $X \rightarrow Y$.
2. Given a metric space X , what is $\text{QI}(X)$?
3. What groups are quasi-isometric to a given metric space X ?
4. How can one recognize when a space is quasi-isometric to X ?

Motivation

- Determine the structure of the outer automorphism group or the commensurator of G .
- Extend Mostow rigidity to other geometries.
- Develop deeper understanding of the intrinsic/asymptotic geometry of a group (or space).
- Develop new geometric, analytic, or combinatorial tools for the associated asymptotic geometry.
- Because it's there.

Thm. (Mostow, Kleiner-Leeb) Suppose X, Y are symmetric spaces of noncompact type. Then X is quasi-isometric to Y iff X is isometric to Y (modulo normalization of factor metrics).

Thm. (Pansu, Schwartz, Eskin-Farb, Eskin) Suppose $G \subset \text{Isom}(X)$ and $G' \subset \text{Isom}(X')$ are irreducible nonuniform lattices, where X and X' are symmetric spaces of noncompact type. Then G is quasi-isometric to G' iff G is commensurable to G' .

Fact. Nobody understands $\text{QI}(\mathbb{R}^n)$, when $n \geq 2$.

Thm. When $n \geq 3$, $\text{QI}(\mathbb{H}^n)$ is canonically isomorphic to the group of quasiconformal homeomorphisms of the standard sphere S^{n-1} .

Thm (Pansu, Kleiner-Leeb, Bourdon-Pajot, Bourdon, Xie) Let $X = \prod_i X_i$ be a product, where each X_i is either

- An irreducible symmetric space of non-compact type other than a hyperbolic or complex hyperbolic space.
- A thick irreducible Euclidean building.
- A Fuchsian building.

Then (modulo renormalizing factor metrics), every quasi-isometry $X \rightarrow X$ is equivalent to a product of isometries.

Thm. (Eskin-Fisher-Whyte) The quasi-isometry group of three-dimensional Solv geometry is, up to index two, isomorphic to $\text{BiLip}(\mathbb{R}) \times \text{BiLip}(\mathbb{R})$.

Thm. (Stallings, Dunwoody, Gromov, Karrass-Petrowski-Solitar) Any group quasi-isometric to a tree is virtually free.

Thm. (Gromov, Pansu, Bass) Any group quasi-isometric to \mathbb{R}^n is virtually \mathbb{Z}^n .

Thm. (Sullivan, Gromov, Tukia, Pansu, Gabai, Casson-Jungreis, Hinkannen, Cannon-Swenson, Kleiner-Leeb) Suppose X is a symmetric space of noncompact type. Then any group G quasi-isometric to X admits a discrete, cocompact, isometric action on X .

Thm. (Schwartz, Eskin-Farb, Eskin) Suppose G is an irreducible nonuniform lattice in $\text{Isom}(X)$, where X is a symmetric space of noncompact type. Then any group quasi-isometric to G is commensurable to G .

Thm. (Farb-Mosher) A group quasi-isometric to a Baumslag-Solitar group $BS(1, m)$ is commensurable to it.

Thm. (Eskin-Fisher-Whyte, Dymarz) Let M be a symmetric $n \times n$ matrix with no eigenvalues on the unit circle. A group quasi-isometric to semi-direct product $\mathbb{R} \rtimes_M \mathbb{R}^n$ is virtually a lattice in the semi-direct product $\mathbb{R} \rtimes_{M^\alpha} \mathbb{R}^n$ for some $\alpha \neq 0$.

Recognition

There are recognition theorems for

- \mathbb{R} .
- The trivalent tree.
- (Tukia-Vaisala, Bonk-Schramm) \mathbb{H}^2 .
- (Cannon, Bonk-Kleiner) \mathbb{H}^3 .

Recognition of \mathbb{R}^2 is wide open.

Outlook

Conj. If G is a generic finitely generated group, then the natural map $G \rightarrow \text{QI}(G)$ is an isomorphism.

Prob. The quasi-isometry classification of nilpotent groups is wide open.

Other classes of groups:

Solvable groups, Coxeter groups, buildings, Artin groups, Gromov-Thurston examples, ...

Boundaries $CAT(0)$ and $CAT(-1)$ spaces.

Let X be a $CAT(0)$ space.

Given two constant speed geodesic rays $\gamma_1 : [0, \infty) \rightarrow X$, $\gamma_2 : [0, \infty) \rightarrow X$, define their (asymptotic) distance to be

$$\lim_{t \rightarrow \infty} \frac{d(\gamma_1(t), \gamma_2(t))}{t}.$$

This is a pseudo-metric on the collection of geodesic rays; the quotient metric space is the **Tits cone** $C_T X$.

Given $p \in X$, the inclusion of the set of rays starting at p maps onto $C_T X$.

Examples. \mathbb{R}^n . \mathbb{H}^n . $\mathbb{C}H^m$, $\mathbb{H}^k \times \mathbb{H}^l$. Symmetric spaces.

This Tits cone isometric to a Euclidean cone over another metric space, the **Tits boundary** $\partial_T X$.

Alternate definition of $\partial_T X$. Given $p, x, y \in X$, we denote by $\tilde{\angle}_p(x, y)$ the (Euclidean) comparison angle between x and y at p .

If γ_1, γ_2 are rays leaving p , then

$$\begin{aligned} & \angle_T(\gamma_1, \gamma_2) \\ & := \lim_{R \rightarrow \infty} \{ \tilde{\angle}_p(x_1, x_2) \mid x_i \in \gamma_i, d(x_i, p) > R \} \end{aligned}$$

The isometry group of X acts isometrically on $\partial_T X$.

Suppose X is $CAT(-1)$. Then $C_T X$ is a cone over a discrete space, which isn't very interesting.

Pick $p \in X$. Repeat the definition of the Tits boundary, but using the **hyperbolic** comparison angle:

$$\tilde{\angle}_p^{\mathbb{H}^2}(x, y),$$

which is defined using comparison triangles in \mathbb{H}^2 .

The metric one obtains depends on the basepoint, but only up to quasiconformal, or quasimobius homeomorphism.

Examples. \mathbb{H}^k . $\lambda\mathbb{H}^k$. $\mathbb{C}\mathbb{H}^k$.

Relation between X and ∂X :

- Complete geodesics in X are in bijection with pairs of points in ∂X .
- Triples in ∂X determine points in X , up to uniformly bounded error. If $\text{Isom}(X)$ acts cocompactly, the converse is true.

$\text{Isom}(X)$ does not act by isometries, but by quasiconformal, or quasiMöbius homeomorphisms.

Boundaries of Gromov hyperbolic spaces

Def. A geodesic metric space X is δ -hyperbolic if every geodesic triangle in X is δ -thin.

Convention. Unless otherwise indicated, our Gromov hyperbolic spaces will be proper.

Suppose X is a proper δ -hyperbolic space, and $p \in X$.

The **Gromov overlap** of $x, y \in X$ is

$$\langle x | y \rangle := \frac{1}{2} (d(p, x) + d(p, y) - d(x, y)).$$

If $\gamma_1, \gamma_2 \subset X$ are geodesic rays leaving p , then their overlap is

$$\langle \gamma_1 | \gamma_2 \rangle$$

$$:= \lim_{R \rightarrow \infty} \{ \langle x_1, x_2 \rangle \mid x_i \in \gamma_i, d(x_i, p) > R \}$$

Visual metrics. A **visual metric** on ∂X with parameter $c > 0$ is a (pseudo)-distance d on the collection of rays leaving p such that

$$d(\gamma_1, \gamma_2) \sim e^{-c} \langle \gamma_1 | \gamma_2 \rangle.$$

When c is sufficiently small, visual metrics always exist.

Thm. Every (L, A) -quasi-isometry

$$f : X \rightarrow Y$$

between δ -hyperbolic spaces induces an η -quasi-Mobius homeomorphism

$$\partial f : \partial X \rightarrow \partial Y,$$

where $\eta = \eta(L, A, \delta)$.

If X and Y have cocompact isometry groups, the converse is true. In particular we have an isomorphism

$$\text{QI}(X) \simeq \text{QM}(X) \subset \text{Homeo}(\partial X).$$

Quasi-Mobius homeomorphisms

Def. Let Z be a metric space. The **cross-ratio** of a quadruple $(x, y, z, w) \in Z^4$ is

$$[x, y, z, w] := \frac{d(x, z)d(y, w)}{d(x, w)d(y, z)}.$$

Def. Suppose $\eta : [0, \infty) \rightarrow [0, \infty)$ is a homeomorphism. Then a homeomorphism $f : X \rightarrow Y$ between metric spaces is η -quasi-Mobius if for every quadruple (x, y, z, w)

$$[f(x), f(y), f(z), f(w)] \leq \eta([x, y, z, w]).$$