HOMOLOGY STABILITY FOR $\mathcal{O}_{n,n}$

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INTRODUCTION

In algebraic $K$-theory it is useful to know whether the sequence of homomorphisms

$$\ldots \to H_k(\mathcal{O}_{n,n}(F)) \xrightarrow{i*} H_k(\mathcal{O}_{n+1,n+1}(F)) \to \ldots$$

eventually stabilizes to a sequence of isomorphisms. An unpublished result of Daniel Quillen states that a similar sequence for $\text{GL}_n(F)$ does stabilize. In this paper we adapt Quillen's method to $\mathcal{O}_{n,n}(F)$, where $F$ is a field of characteristic zero.
If $G$ is a group, we let $H_\ast(G)$ denote group homology with coefficients in $\mathbb{Z}$; $\varepsilon_{n,n}^0(F)$ denotes the orthogonal group of the quadratic form $\begin{pmatrix} 0 & I \\ \varepsilon I & 0 \end{pmatrix}$ on a $2n$-dimensional vector space over $F$ ($\varepsilon = \pm 1$, and $I$ is the $n \times n$ identity matrix).

**Theorem.** If $\text{char } F = 0$, then the homomorphism

$$i_* : H_k(\varepsilon_{n,n}^0(F)) \rightarrow H_k(\varepsilon_{n+1,n+1}^0(F))$$

is onto for $n \geq 3k+1$ and an isomorphism for $n \geq 3k+3$.

We associate to $\varepsilon_{n,n}^0(F)$ a building-like simplicial complex $X$. The action of $\varepsilon_{n,n}^0(F)$ on $X$ gives rise to a spectral sequence converging to zero which gives information about the homology of $\varepsilon_{n,n}^0(F)$. The theorem is proved by comparing the spectral sequences for $\varepsilon_{n,n}^0(F)$ and $\varepsilon_{n+1,n+1}^0(F)$ (or, more precisely, by constructing a relative spectral sequence giving information about the relative homology.)

In the first section we define the complex $X$ and prove that it is homotopy equivalent to a wedge of spheres. In the second section we construct the spectral sequence and prove the theorem.
1. **The building-like simplicial complex associated to** $\mathfrak{0}_{n,n}^n(F)$.

A. In this section, let $F$ be any field, and $V$ a $2n$-dimensional vector space over $F$, with a polar basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ (i.e., in this basis the matrix of the quadratic form is $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where $I$ is the $n$ by $n$ identity matrix). Let $\cdot$ denote the inner product associated to the form.

**Definition.** A subspace $A$ of $V$ is called **totally isotropic** if $v \cdot w = 0$ for all vectors $v$ and $w$ in $A$.

The set of all non-trivial totally isotropic subspaces of $V$ is partially ordered by inclusion. The geometric realization of this partially ordered set is a simplicial complex, which we will call $X$; a $k$-dimensional simplex of $X$ is a chain of $k + 1$ totally isotropic subspaces of $V$

$$A_0 \subset A_1 \subset \ldots \subset A_k.$$ 

By Witt's theorem [5], every maximal totally isotropic subspace of $V$ has dimension $n$; hence our complex $X$ has dimension $n - 1$. 

Proposition 1.1. \( X \) is homotopy equivalent to a wedge of \((n-1)\)-dimensional spheres.

**Proof:** Let \( E = \langle e_1, \ldots, e_n \rangle \) be the subspace of \( V \) spanned by the vectors \( e_i \).

For each \( k = 0, \ldots, n \), define a subcomplex \( X_k \) of \( X \) by

\[
X_k = \text{the union of all maximal simplices } A_1 \subset \ldots \subset A_n \\
\text{such that } \dim(A_n \cap E) \geq n - k.
\]

\[
= \bigcup_{\dim(A_n \cap E) > n-k} \text{st } A_n.
\]

Thus \( X_0 \) is just the closed star of \( E \), \( X_n \) is \( X \), and we have inclusions

\[
X_0 \subset X_1 \subset \ldots \subset X_n.
\]

**Claim.** \( X_{k-1} \) is a deformation retract of \( X_k \) for \( 1 \leq k \leq n - 1 \).

Assuming the claim, we have \( X_{n-1} \) contractible (since \( X_0 = \text{st} E \) is contractible). \( X_{n-1} \) is the entire complex minus the stars of maximal isotropic subspaces \( A_n \) such that \( A_n \cap E = 0 \). Since for such \( A_n \), \( \text{lk } A_n \subset X_{n-1} \), by contracting \( X_{n-1} \) to a point we obtain
$$X \simeq \bigvee_{A_n \cap E = 0} \text{susp}(\text{lk } A_n) .$$

($\simeq$ denotes homotopy equivalence)

But $\text{lk } A_n$ is just the Tits building for $A_n$ (since every subspace of a totally isotropic space is clearly totally isotropic). Therefore by the Solomon-Tits theorem [3], $\text{lk } A_n$ is homotopy equivalent to a wedge of $(n-2)$-dimensional spheres.

So

$$X \simeq \bigvee \text{susp}(\bigvee_{S^{n-2}}) \simeq \bigvee S^{n-1} ,$$

and we are done.

Proof of claim. We do this in two steps.

Step 1. Let $A$ and $A'$ be two different maximal isotropic subspaces with $\dim(A \cap E) = \dim(A' \cap E) = n - k$ , (i.e., $A, A' \subset X_k \setminus X_{k-1}$) . Then

$$\text{st } A \cap \text{st } A' \subset X_{k-1} .$$

Proof. A simplex $\sigma$ in $\text{st } A \cap \text{st } A'$ looks like

$$\sigma = A_0 \subset \ldots \subset A_s ,$$

with $A_s \subset A \cap A'$ . So to show $\sigma \in X_{k-1}$, it is sufficient to find a maximal isotropic subspace $B$ with $A \cap A' \subset B$ and $\dim(B \cap E) \geq n-k+1$ (i.e., $B \subset X_{k-1}$) .

Let $r = \dim(A \cap A' \cap E)$ . Then $r \leq n - k$ (*).

Since $\dim(A \cap E) = n - k$ .
Let $r + s = \dim(A \cap A')$. Then $r + s < n(**)$ since $A \neq A'$.

We can find a basis \{${u_1, \ldots, u_r, v_1, \ldots, v_s}$\} for $A \cap A'$ with the $u_i$ in $E$ and $v_i$ not in $E$. Since $\dim(<v_1, \ldots, v_s^* \cap E) \geq n - s$, we can add $n - s - r$ independent vectors in $E$ to our basis to obtain a maximal isotropic subspace $B$ containing $A \cap A'$. Then $\dim(B \cap E) = (n - s - r) + r = n - s$.

If $r = n - k$, then (**) implies $n - s > r$, so $n - s > n - k$ and we are done.

If $r < n - k$, we need to estimate $s$:

We notice that the subspace

$$(A \cap E) + (A' \cap E) + (A \cap A')$$

is totally isotropic and hence has dimension $\leq n$; by counting dimensions we have

$$(n - k) + (n - k) + (s + r) - 2r \leq n,$$

or $s \leq 2k + r - n$.

Then $n - s \geq n - (2k + r - n) = 2n - 2k - r > 2n - 2k - (n - k) = n - k$

and we are done.

**Step 2.** Let $A$ be maximal isotropic with $\dim(A \cap E) = n - k$. Then $\overline{\text{st} \, A \cap X_{k-1}}$ is a deformation retract of $\overline{\text{st} \, A}$. 
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Proof. First note that $\overline{\text{st} A} \cap X_{k-1} \neq \emptyset$, since $E' = A \cap E \neq 0$. Let $A_s$ be a vertex in $\overline{\text{st} A} \cap X_{k-1}$, and consider the subspace $A_s + E'$. Clearly $A_s + E' \subset A$. In fact, $A_s + E' \subset \not A$, since if $A_s + E' = A$, then there is a $k$-dimensional subspace of $A_s$ which does not intersect $E$, contradicting the assumption that $A_s \subset X_{k-1}$.

We can write $A_s + E'$ as $A_r \oplus E'$, where $A_r \cap E \subset A_r \cap E' = 0$. Now $\dim [(A_r \oplus E')^\perp \cap E] = \dim (A_r^\perp \cap E) = n - r$. Since $\dim (A_r \oplus E') = r + n - k \leq n - 1$, we have $n - r \geq n - k + 1$, which implies that $A_r \oplus E' = A_s + E' \subset X_{k-1} \cap \overline{\text{st} A}$.

Now the simplicial maps

$$
\begin{align*}
A_1 & \subset A_2 \subset \cdots \subset A_s \\
\downarrow & \\
A_1 + E' & \subset A_2 + E' \subset \cdots \subset A_s + E' \\
\downarrow & \\
E' & 
\end{align*}
$$

induce a contraction of $\overline{\text{st} A} \cap X_{k-1}$ to the vertex $E'$.

Since $\overline{\text{st} A}$ is itself contractible, we have $\overline{\text{st} A} \cap X_{k-1}$ is a contractible subcomplex of a contractible complex, so there is a deformation retraction of $\overline{\text{st} A}$ onto $\overline{\text{st} A} \cap X_{k-1}$.

Combining steps 1 and 2, we see that there is a retraction of $X_k$ to $X_{k-1}$, and we are done. $\square$
B. Now consider a different filtration of \( X \), namely, let

\[
Y_i = \text{the union of all closed } i\text{-dimensional simplices of the form } A_1 \subset \ldots \subset A_{i+1},
\]

with \( \dim A_j = j \).

Then we have \( \phi \subset Y_0 \subset Y_1 \subset \ldots \subset Y_{n-1} = X \).

Lemma 1.2. 1) \( H_i(Y_j) = 0 \) for \( 1 \leq i < j \leq n-1 \)

2) \( H_0(Y_j) \to H_0(X) \) is onto for \( j = 0 \)

and an isomorphism for \( j > 0 \).

Proof: This follows from Lemma 1.6 of the book *The Discrete Series Representations of GL_n* of a Finite Field, by George Lusztig ([6]), if we use Proposition 1.1 and the Solomon-Tits theorem. □

Lemma 1.3. The sequence

\[
0 \to H_{n-1}(X) \to H_{n-1}(X, Y_{n-2}) \to H_{n-2}(Y_{n-2}, Y_{n-3}) \to \ldots
\]

\[
\ldots \to H_2(Y_2, Y_1) \to H_1(Y_1, Y_0) \to H_0(Y_0) \to H_0(X) \to 0
\]

is exact, where the maps are the usual boundary maps \( \partial \).

We give a proof of this lemma (Lemma 1.1 of Lusztig's book) even though Lusztig omitted it, since it is not true exactly as Lusztig stated it.

Proof: Associated to the filtration \( \phi \subset Y_0 \subset \ldots \subset Y_{n-1} = X \) is a spectral sequence with
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\[ E^1_{p,q} = H_p(Y_p, Y_{p-1}) \Rightarrow H_{p+q}(X) . \]

Since $Y_p$ is a p-dimensional complex, $E^1_{p,q} = 0$ for $q > 0$.

By Lemma 1.2, $E^1_{p,q} = 0$ for $q < 0$ also. Since the sequence converges to $H(X)$, which is zero except in degree zero and degree $n - 1$, the line $q = 0$ is exact except at $H_{n-1}(X, Y_{n-2})$ and at $H_0(Y_0)$.

We calculate

\[ \text{coker}(\beta: H_1(Y_1, Y_0) \to H_0(Y_0)) \cong H_0(Y_1) \cong H_0(X) \]

and

\[ \text{ker}(\beta: H_{n-1}(X, Y_{n-2}) \to H_{n-2}(Y_{n-2}, Y_{n-3})) \cong H_{n-1}(X, Y_{n-3}) \cong H_{n-1}(X) . \]

We now have the exact sequence in the statement of the lemma. \(\square\)

**Lemma 1.4.** For $i \geq 2$, $H_i(Y_i, Y_{i-1}) \cong \bigoplus_{A \subseteq V} H_{i-1}(T(A)) , \quad \text{dim } A = i+1$

A totally isotropic

where $T(A)$ is the Tits building of $A$.

Also $H_1(Y_1, Y_0) \cong \bigoplus_{A \subseteq V} \text{ker}(H_0(T(A)) \to \mathbb{Z}) , \quad \text{dim } A = 2$

A totally isotropic
and \( H_0(Y_0) \cong \emptyset \). 

\[ <v> \subset V \]

\( v \) isotropic

**Proof:** This is a translation of p.11, [6] to our case. \( \Box \)

If we let \( \tau(A) = H_{i-1}(T(A)) \), the exact sequence of lemma 1.3 becomes

\[
0 \rightarrow H_{n-1}(X) \rightarrow \emptyset \rightarrow A \subset V \rightarrow \emptyset \rightarrow \tau(A) \rightarrow \ldots \rightarrow A \subset V \rightarrow \emptyset \rightarrow \tau(A) \rightarrow 0
\]

\[
\text{dim } A = n \quad \text{dim } A = 3
\]

A totally isotropic \quad A totally isotropic

\[
\rightarrow \emptyset \rightarrow \ker(\tau(A) \rightarrow Z) \rightarrow \emptyset 0
\]

\[
\text{dim } A = 2 \quad \text{dim } A = 1
\]

a totally isotropic \quad A isotropic

(1)

\[
0_{n,n}(F) \text{ acts transitively on this acyclic complex.}
\]

2. **Proof of the theorem in the case** \( \text{char } F = 0 \).

A. We want to prove that \( i_* : H_k(0_{n,n}(F)) \rightarrow H_k(0_{n+1,n+1}(F)) \) is onto for \( n \geq 3k + 1 \) and an isomorphism for \( n \geq 3k + 3 \). Actually, we will prove that the relative homology groups vanish for \( n \) large enough with respect to \( k \). To do this, we want to produce a spectral sequence involving these groups.

**Notation.** Let \( G_n \) denote \( 0_{n,n}(F) \).
Also, let $K_i = \bigoplus A \subset V \tau(A)$ for $2 < i < n$; let $K_0 = Z$, $\dim A = i$.

$K_1 = \bigoplus Z$, $K_2 = \bigoplus \ker(\tau(A) \to Z)$ and $A \subset V$, $\dim A = 1$; $A$ isotropic.

$K_{n+1} = H_{n-1}(X)$. In other words, we denote the acyclic complex (1) produced in the last section by

$$0 \to K_{n+1} \to K_n \to \ldots \to K_0 \to 0.$$ 

Let $E_{G_n}$ be a $G_n$-free resolution of $Z$. We form the standard double complex $K_i \otimes_{G_n} E^j G_n$ by taking maps

$$
\begin{array}{ccc}
K_i & \otimes & E_{j+1} G_n \quad \downarrow \quad \otimes d \quad \downarrow (-1)^{i+1} \triangledown 1 \\
\downarrow \quad \otimes (-1)^j \gamma \triangledown 1 & \quad \otimes d \quad \downarrow (-1)^j \triangledown 1 \\
K_{i+1} & \otimes & E_{j+1} G_n
\end{array}
$$

where $\triangledown$ is the differential in the complex $K$ and $d$ is the differential in $E_{G_n}$.

This double complex gives a single complex if we let

$$A_p = \bigoplus_{i+j=p} K_i \otimes E^j G_n$$
with differential $\delta : A^p + A^{p-1}$ defined by
\[ \delta = 1 \circ d + \pm 3 \circ d . \]

We can filter this single complex in two different ways; namely, we take the filtrations of $A$ induced by the horizontal and vertical filtrations of the double complex $K_i \otimes E \mathbb{Z}_n$.

These two filtrations give us two different spectral sequences both converging to the homology of $A$. The vertical filtration gives
\[ E^1_{p,q} = H_q(K_* \otimes E \otimes \mathbb{Z}_n, \partial \partial 1) \]
which is identically zero since the sequence $\{K_p\}$ is exact.

The horizontal filtration gives the spectral sequence
\[ E^1_{p,q} = H_q(K_p \otimes E_i \otimes \mathbb{Z}_n, 1 \circ d) \]
\[ = H_q(G_n; K_p) \]
This spectral sequence must converge to zero since the vertical one does. We will now calculate the $E^1$ terms of this sequence.

Recall that
\[ K_p = \otimes \tau(A) . \]
\[ \dim A = p \]
\[ A \text{ tot. isot. } < V \]
Since $G_n$ acts transitively on $p$-dimensional totally isotropic subspaces of $V$, this is
\[ = \mathbb{Z}[G_n] \otimes \mathbb{Z}[\text{stabilizer of } \langle e_1, \ldots, e_p \rangle] \tau(F^p) \]

If we denote the stabilizer of \( \langle e_1, \ldots, e_p \rangle \) by \( S_{p,n} \), we have

\[ E_{p,q}^1 = H_q(G_n; K_p) = H_q(G_n; \mathbb{Z}[G_n] \otimes \mathbb{Z}[S_{p,n}] \tau(F^p) \]

which, by Shapiro's lemma [7] is

\[ = H_q(S_{p,n}; \tau(F^p)) \cdot \]

It is not hard to calculate the stabilizer subgroup \( S_{p,n} \); it turns out to be (conjugate to the group of) all matrices of the form

\[
\begin{pmatrix}
\alpha & * & * \\
0 & A & * \\
0 & 0 & t_{\alpha^{-1}}
\end{pmatrix}
\]

where \( \alpha \) is any matrix in \( \text{GL}_p(F) \), \( A \) is any matrix in \( G_{n-p} \), and there are conditions on the \( * \) terms to insure that the whole matrix lies in \( G_n \). (We have conjugated the quadratic form by the matrix

\[
\begin{pmatrix}
I_p & 0 & 0 & 0 \\
0 & I_{n-p} & 0 & 0 \\
0 & 0 & 0 & I_p \\
0 & 0 & I_{n-p} & 0
\end{pmatrix}
\]

in order to simplify writing down the matrices; this does not change the homology.)
We have an obvious inclusion

\[ i : (\text{GL}_p \times \text{G}_{n-p}) \to \text{S}_{p,n} \, . \]

**Proposition 2.1.** If \( n \geq 3t + 1 \), then \( i_* : H_t(\text{GL}_p \times \text{G}_{n-p}) \to H_t(\text{S}_{p,n}) \) is an isomorphism for \( 0 \leq t \leq k \).

The proposition will be proved in part B. of this section. It should be noted here, however, that the proof of the proposition depends on the fact that char \( F = 0 \).

In our spectral sequence (*) we now have

\[ E^{1}_{p,q} = H_q(\text{GL}_p \times \text{G}_{n-p}; \tau(F^p)) \]

if we choose \( n \geq 3q + 1 \).

Since we are interested in the relative homology groups, we actually want a "relative" spectral sequence instead of (*). This is constructed in the following way.

The inclusion \( i : \text{G}_n \to \text{G}_{n+1} \) induces an equivariant chain map from the complex \( \{K_i\} \) associated to \( \text{G}_n \) to the analogous complex \( \{K'_i\} \) associated to \( \text{G}_{n+1} \).

We form a double complex as follows:
In this double complex, the columns are just the mapping cones of the chain maps

$$i_* : K_s \otimes E_*G_n \to K'_s \otimes E_*G_{n+1}.$$ 

As before, we see that the horizontal filtration gives a spectral sequence which is identically zero since the sequences \(\{K_i\}\) and \(\{K'_i\}\) are exact. The vertical filtration gives a spectral sequence with

$$E^1_{s,t} = H_t$$ (mapping cone of

$$i_* : K_s \otimes E_*G \to K'_s \otimes E_*G_{n+1}$$)

which must also converge to zero.

We know from computing the terms of the spectral sequence (*) that

$$\tau(F^S) \otimes_{S,n} E^*S_{s,n} \to K_s \otimes_{G_n} E^n \to$$

and

$$\tau(F^S) \otimes_{S,n+1} E^*_S, n+1 \to K'_s \otimes_{G_{n+1}} E^n_{n+1}.$$
are homology equivalences, so in (**),

\[ E^1_{s,t} = H_t \text{ (mapping cone of) } \]

\[ i_*: \tau(F^S) \otimes E^*S_{s,n} \to \tau(F^S) \otimes E^*S_{s,n+1} \]

\[ = H_t(S_{s,n+1}, S_{s,n} : \tau(F^S)) \]

which, by Proposition 2.1 is

\[ = H_t(GL_s \times (G_{n+1-s}, G_{n-s}); \tau(F^S)) \]

If \( n \geq 3k + 1 \).

We now have our relative spectral sequence, and will prove the theorem by induction on \( k \). That is, we will assume that for \( q \leq k - 1 \) and \( n \geq 3q + 1 \),

\[ H_q(G_{n+1}, G_n) = 0 \]

Using the universal coefficient theorem and the Kunneth formula, we calculate

\[ H_t(GL_s \times (G_{n+1-s}, G_{n-s}); \tau(F^S)) \]

\[ = \bigoplus_{p+q=t} H_p(GL_s) \otimes H_q(G_{n+1-s}, G_{n-s}) \otimes \tau(F^S) \]

\[ \bigoplus_{p+q=t-1} \text{Tor}(H_p(GL_s), H_q(G_{n+1-s}, G_{n-s})) \otimes \tau(F^S) \]

\[ \bigoplus \text{Tor}(p+q=t-1, H_p(GL_s) \otimes H_q(G_{n+1-s}, G_{n-s}), \tau(F^S)) \]

\[ \bigoplus \text{Tor}(p+q=t-2, \text{Tor}(H_p(GL_s), H_q(G_{n+1-s}, G_{n-s})), \tau(F^S)) \]

By our induction assumption, all the terms \( H_q(G_{n+1-s}, G_{n-s}) \) vanish for \( q \leq k - 1 \) and \( n - s \geq 3q + 1 \). If we assume now that \( n \geq 3k \), it is easily seen that
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$$E^1_{s,t} = H_t(\text{GL}_s \times (G_{n+1-s}, G_{n-s}); \tau(F^S)) = 0$$

for $t = 0, \ldots, k - 1$ and $s \leq k + l - t$.

We also have

$$E^1_{0,k} = H_k(G_{n+1}, G_n; \mathbb{Z})$$

and

$$E^1_{l,k} = H_k(\text{GL}_l \times (G_n, G_{n-1}); \mathbb{Z}) = H_0(\text{GL}_l; \mathbb{Z}) \otimes H_k(G_n, G_{n-1}; \mathbb{Z})$$

$$= H_k(G_n, G_{n-1}; \mathbb{Z})$$

since $H_0(\text{GL}_l; \mathbb{Z}) = \mathbb{Z}/\{n(g - 1) : n \in \mathbb{Z}, g \in \text{GL}_l\} = \mathbb{Z}$.

Now the $E^1$ level of the spectral sequence looks like

\[
\begin{array}{cccccccc}
  k & H_k(G_{n+1}, G_n) & \overset{d_1}{\longrightarrow} & H_k(G_n, G_{n-1}) & \leftarrow & * & \leftarrow & * \\
  k-1 & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & * \\
  . & . & \leftarrow & . & \leftarrow & . & \leftarrow & * \\
  . & . & \leftarrow & . & \leftarrow & . & \leftarrow & * \\
  . & . & \leftarrow & . & \leftarrow & . & \leftarrow & * \\
  1 & & & & & & & 0 & * \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\end{array}
\]

Since $E^1_{s,k-(s-1)} = 0$ for $s = 2, \ldots, k + 1$, the
differentials $d_s : E^s_{s,k-(s-1)} \rightarrow E^s_{0,k}$ are zero. Thus

$E^\infty_{0,k} = E^1_{0,k}/\text{im} d_1$. But we know $E^\infty_{0,k} = 0$ since the
spectral sequence converges to zero, so we must have
\[ E^1_{0,k} = \text{im } d_1, \text{ i.e. the map } \]
\[ d_1 : H_k(G'_n, G'_{n-1}) \to H_k(G'_{n+1}, G'_n) \]
is onto for \( n \geq 3k \).

We now consider the following diagram

\[ \begin{array}{ccccccccc}
H_k(G'_n, G'_{n-1}) & \to & H_k(G'_{n-1}, G'_{n-1}) & \to & H_k(G'_{n-1}, G'_n) & \to & H_k(G'_{n}, G'_{n-1}) \\
\downarrow d_1 & & \downarrow & & \downarrow & & \downarrow \\
H_k(G'_{n+1}, G'_n) & \to & H_k(G'_n, G'_n) & \to & H_k(G'_n, G'_{n+1}) & \to & H_k(G'_{n+1}, G'_n) \\
\downarrow & & \downarrow i_* & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array} \]

Our induction assumption implies \( H_k(G'_{n+1}, G'_n) = H_k(G'_n, G'_{n-1}) = 0 \) (since we assumed that \( n \geq 3k \)).

Thus \( i_* \) is surjective. A diagram chase shows that \( i_* \) is also injective.

We can then construct the diagram

\[ \begin{array}{ccccccccc}
H_k(G'_n, G'_{n-1}) & \to & H_k(G'_{n+1}, G'_n) & \to & H_k(G'_n, G'_{n+1}) & \to & H_k(G'_{n+1}, G'_n) & \to & H_k(G'_{n+2}, G'_n) \\
\downarrow & & \downarrow d_1 & & \downarrow & & \downarrow & & \downarrow \\
H_k(G'_n, G'_{n+1}) & \to & H_k(G'_{n+2}, G'_{n+1}) & \to & H_k(G'_{n+1}, G'_{n+1}) & \to & H_k(G'_{n+1}, G'_n) & \to & H_k(G'_{n+2}, G'_n) \\
\downarrow & & \downarrow j_* & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array} \]

where \( i_* \) and \( j_* \) are both isomorphisms. Another diagram chase shows that \( H_k(G'_{n+2}, G'_n) = 0 \) (\( n \geq 3k \)).
Thus we have shown that $H_k(G_{n+1}, G_n) = 0$ for $n \geq 3k + 1$, completing the induction step. Note that we needed only $n \geq 3k = 3(k - 1) + 3$ to prove that $i_* : H_{k-1}(G_n) \to H_{k-1}(G_{n+1})$ is an isomorphism; if we repeat the argument for $k+1$ instead of $k$, we see that

$$i_* : H_k(G_n) \to H_k(G_{n+1})$$

is an isomorphism for $n \geq 3k + 3$. □

B. We will now prove Proposition 2.1. Actually, we will prove a stronger statement than what we need, namely

**Proposition 2.2.** Let $F$ be a field of characteristic zero and $S_{p,n}$ and $G_n$ as above. Then for all $p$, $n$ and $t$,

$$i_* : H_t(GL_p \times G_{n-p}) \to H_t(S_{p,n})$$

is an isomorphism.

This proposition depends on the fact that $\text{char } F = 0$. It is not true, for example, for a finite field, since then the groups involved are finite groups, and a map inducing an isomorphism on the homology of two finite groups is itself an isomorphism [2].

If we can prove this proposition for homology with
coefficients in any algebraically closed field $k$, it will be true for homology with coefficients in $\mathbb{Z}$ by the following standard argument.

**Lemma 2.3.** If $H_*(A; k) = 0$ for any algebraically closed field $k$, then $H_*(A; \mathbb{Z}) = 0$.

**Proof:** Suppose char $k = 0$. The universal coefficient theorem gives

$$H_*(A; k) \cong H_*(A; \mathbb{Q}) \otimes_{\mathbb{Q}} k \oplus \text{Tor}(H_*(A; \mathbb{Q}), k).$$

The torsion term is zero since $k$ is a free $\mathbb{Q}$-module. Thus $H_*(A; k) = 0$, if and only if $H_*(A; \mathbb{Q}) = 0$.

If char $k = p$, we have

$$H_*(A; k) \cong H_*(A; \mathbb{Q}/p) \otimes_{\mathbb{Q}/p} k \oplus \text{Tor}(H_*(A; \mathbb{Z}/p), k).$$

Here, too, $H_*(A; k) = 0$ implies that $H_*(A; \mathbb{Z}/p) = 0$.

Now consider the Bockstein homology sequence for the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

which is

$$\ldots \to H_n(A; \mathbb{Q}/\mathbb{Z}) \to H_{n-1}(A; \mathbb{Z}) \to H_{n-1}(A; \mathbb{Q}) \to \ldots$$

Since $H_{n-1}(A; \mathbb{Q}) = 0$, we have $H_n(A; \mathbb{Q}/\mathbb{Z}) \cong H_{n-1}(A; \mathbb{Z})$.

Now $\mathbb{Q}/\mathbb{Z}$ is a torsion group; it is the direct limit

$$\mathbb{Q}/\mathbb{Z} = \lim_{\longleftarrow} \mathbb{Z}/p^n.$$
Since homology commutes with direct limits, we need only show that $H_n(A; \mathbb{Z}/p^n) = 0$ for all $p$ and $n$. To do this, we consider the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^n \to 0$$

which gives the homology sequence

$$\ldots \to H_{i+1}(\mathbb{Z}/p^{n-1}) \to H_i(\mathbb{Z}/p) \to H_i(\mathbb{Z}/p^n) \to H_i(\mathbb{Z}/p^{n-1}) \to \ldots$$

By induction we may assume $H_n(\mathbb{Z}/p^{n-1}) = H_n(\mathbb{Z}/p) = 0$, which implies $H_n(\mathbb{Z}/p^n) = 0$. □

Corollary. If $\omega : X \to Y$ induces an isomorphism $\omega_* : H_n(X; k) \to H_n(Y; k)$ on homology with coefficients in any algebraically closed field $k$, then $\omega_* : H_n(X; \mathbb{Z}) \to H_n(Y; \mathbb{Z})$ is an isomorphism.

Proof: In Lemma 2.3, let $A$ be the mapping cone of $\omega$. □

Thus we may assume that we have coefficients in an algebraically closed field $k$.

Consider the short exact sequence of subgroups of $G_n$ consisting of matrices of the form

$$I \to \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} \alpha & * & * \\ 0 & A & * \\ 0 & 0 & t^{-1}_\alpha \end{pmatrix} \to \begin{pmatrix} \alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & t^{-1}_\alpha \end{pmatrix} \to I$$
where $A \in G_{n-p}$ and $\alpha \in \text{GL}_p$; i.e. this is

$$1 \rightarrow N \rightarrow S_{p,n} \rightarrow \text{GL}_p \times G_{n-p} \rightarrow 1$$

where $S_{p,n}$ and $G_{n-p}$ are the groups defined in 2A., and $N$ is the kernel of the projection

$S_{p,n} \rightarrow \text{GL}_p \times G_{n-p}$.

This short exact sequence gives a first quadrant spectral sequence (the Lyndon-Hochschild-Serre spectral sequence, [4]) with

$$E^2_{s,t} = H_s(\text{GL}_p \times G_{n-p}; H_t(N)) \Rightarrow H_{s+t}(S_{p,n}) .$$

We will examine this spectral sequence in the following two cases.

**Case 1.** $\text{char } F = 0$ and $\text{char } k = \ell$ (so $\ell$ is invertible in $F$).

We can calculate the homology $H_t(N)$ by looking at the short exact sequence

$$1 \rightarrow [N, N] \rightarrow N \rightarrow N/[N, N] \rightarrow 1 \quad (1)$$

A short calculation shows that $N$ is the set of matrices of the form

$$X = \begin{pmatrix}
1 & a & b & c \\
0 & 1 & 0 & -t_b \\
0 & 0 & 1 & -t_a \\
0 & 0 & 0 & 1
\end{pmatrix}$$
where $a$ and $b$ are any $p \times (n-p)$ matrices and $c$ is a $p \times p$ matrix such that $-(c + t_c) = a^t b + b^t a$.

The inverse of $X$ is

$$X^{-1} = \begin{pmatrix}
1 & -a & -b & t_c \\
0 & 1 & 0 & t_b \\
0 & 0 & 1 & t_a \\
0 & 0 & 0 & 1
\end{pmatrix}$$

and the commutators $XYX^{-1}Y^{-1}$ are matrices of the form

$$\begin{pmatrix}
1 & 0 & 0 & d^{-t}d \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

where $d$ is any $p \times p$ matrix. Thus the commutator subgroup $[N, N]$ is isomorphic to the additive abelian group of $p \times p$ skew-symmetric matrices over $F$, and $N/[N, N]$ is isomorphic to the additive abelian group of $p \times 2(n-p)$ matrices over $F$.

The sequence (1) gives a spectral sequence with

$$E^2_{p,q} = H_p(N/[N, N]; H_q([N, N]))$$

converging to the homology of $N$.

We have the following formula for the homology $H_\ast(G; k)$ of an abelian group $G$ with coefficients in a field $k$. Let $\ell = \text{char } k$ and let $\mathcal{G}$ be the
subgroup of elements of \( G \) annihilated by \( \lambda \) if
\( \lambda > 0 \); if \( \lambda = 0 \), set \( _0G = 0 \). Let \( \Lambda(V) \) and
\( \Gamma(V) \) be the exterior and divided power algebras respectively of a \( k \)-vector space \( V \).

**Lemma 2.4.** \( \Lambda(G \otimes_k k) \otimes_k \Gamma(G \otimes_k k) = H_k(G; k) \).

**Proof:** [1]. \( \square \)

Since \( \lambda \) is invertible in \( F \) and \([N, N]\) is abelian, we can compute

\[
H_q([N, N]; k) = \begin{cases} 
0 & \text{for } q > 0 \\
k & \text{for } q = 0 
\end{cases}
\]

Thus in the spectral sequence (2) we have \( E^2_{p,q} = 0 \)
for \( q > 0 \) and \( E^2_{p,0} = H^p_p(N/[N, N]; k) \). But \( N/[N, N] \)
is also abelian; we can compute

\[
H_p(N/[N, N]; k) = \begin{cases} 
0 & \text{for } q > 0 \\
k & \text{for } q = 0 
\end{cases}
\]

Therefore the entire spectral sequence is zero except
that \( E^2_{0,0} = k \). Thus

\[
H_t(N; k) = \begin{cases} 
0 & \text{for } q > 0 \\
k & \text{for } q = 0 
\end{cases}
\]

We now return to the spectral sequence associated
to the short exact sequence \( 1 \to N \to S_{p,n} \to \text{GL}_p \times \text{GL}_{n-p} \to 1 \).
By the above, we now have
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$$E^2_{s,t} = \begin{cases} 
0 & \text{for } t > 0 \\
H_s(\text{GL}_p \times G_{n-p}) & \text{for } t = 0 
\end{cases}$$

Since this spectral sequence converges to the homology of $S_{p,n}$, we have

$$H_s(\text{GL}_p \times G_{n-p}) = H_s(S_{p,n}) \text{ for all } s.$$  

**Case 2.** char $F = \text{char } k = 0$.

Since char $F = 0$, there is an imbedding of the rational numbers $\mathbb{Q}$ into $F$. We identify an element $d \in \mathbb{Q}^*$ with the matrix

$$D = \begin{pmatrix} 
(d) & 0 & 0 \\
0 & I & 0 \\
0 & 0 & (d^{-1}) 
\end{pmatrix}$$

in $\text{GL}_p \times G_{n-p}$, where $(d)$ is the $p \times p$ diagonal matrix with $(d)_{ii} = d$. We can then define an action of $\mathbb{Q}^*$ on $S_{p,n}$ by $d \cdot A = D^{-1}AD$. We will try to identify the induced action of $\mathbb{Q}^*$ on the spectral sequence with

$$E^2_{s,t} = H_s(\text{GL}_p \times G_{n-p}; H_t(N)) \Rightarrow H_{s+t}(S_{p,n}). \quad (3)$$

The action of $\mathbb{Q}^*$ on $\text{GL}_p \times G_{n-p}$ is trivial since $F$ is commutative.
The action of $Q^*$ on $H_*(N)$ is more complicated. To identify it we again look at the short exact sequence (1) and the resulting spectral sequence (2) with

$$E^2_{p,q} = H_p(N/[N, N]; H_q([N, N])) = H_{p+q}(N).$$

A short computation shows that $Q^*$ acts on $[N, N]$ as multiplication by $d^2$ and on $N/[N, N]$ as multiplication by $d$. Note that $[N, N]$ is in the center of $N$, so the action of $[N, N]$ on $H_*(N)$ induced by conjugation is trivial.

By the universal coefficient theorem, we have

$$H_p(N/[N, N]; H_q([N, N])) = H_p(N/[N, N]) \otimes H_q([N, N])$$

(the torsion is zero since we have coefficients in a field).

By the formula for the homology of an abelian group (Lemma 2.4), this is equal to

$$\Lambda^p(N/[N, N] \otimes k) \otimes \Lambda^q([N, N] \otimes k).$$

Thus $d \in Q^*$ acts on the $E^2_{p,q}$ term as multiplication by $d^{p+2q}$. The action commutes with the differentials, so the action on $E^\infty_{p,q}$ is multiplication by $d^{p+2q}$. Thus on the filtration of $H_*(N)$ we have $d$ acting as multiplication by the following powers of $d$:
Returning to the spectral sequence (3), we note that the action of $\mathbb{Q}^*$ on $E^2_{s,t}$ is just the action on $H_t(N)$, since the action on $GL_p \times G_{n-p}$ is trivial. The action again commutes with the differentials, so the action on $E^\infty_{s,t}$ is the same as the action on $E^2_{s,t}$, i.e. the action on $H_t(N)$. Thus in the filtration of $H_r(S_p,n)$ with

$$E^\infty_{k,r-k} = H_i,t-i/H_{i-l},t-i+1$$

we have $d \in \mathbb{Q}^*$ acting by multiplication by the following powers of $d$:

$$\frac{d^{2(r-k)}}{d^{r-k+1}} \frac{d^{r-k}}{}$$

Since $d^m \neq 1$ for $d \neq 1$ or $-1$ and $m > 0$, the action of $\mathbb{Q}^*$ on $H_{k,r-k}$ is not trivial unless $r-k = 0$. However, we know that the action of $\mathbb{Q}^*$ on $H_r(S_p,n) = H_r,0$ is trivial since $\mathbb{Q}^*$ acts on $S_p,n$. 

$$\frac{H_{0,r}}{\cdots} \quad \frac{H_{k,r-k}}{\cdots} \quad \frac{H_{r-1,l}}{\cdots} \quad \frac{H_r,0 = H_r(S_p,n)}{\cdots}$$
by inner automorphism. Therefore we must have

$$H_{k, r-k} = 0 \text{ for } r - k > 0,$$

which means \( E_{s, t}^\infty = 0 \text{ for } t > 0 \) and \( E_{s, 0}^\infty = H_s(S_p, n) \). Since this is a first quadrant spectral sequence, the map \( E_{s, 0}^\infty \rightarrow E_{s, 0}^2 \) is injective, i.e.

$$1 \rightarrow H_s(S_p, n) \xrightarrow{\pi_*} H_s(GL_p \times G_{n-p}).$$

But this map splits, so \( \pi_* \) is also onto, i.e.

$$\pi_* : H_s(S_p, n) \xrightarrow{\cong} H_s(GL_p \times G_{n-p}).$$

**Remark.** (The symplectic case). If instead of

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

we take the form \( \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \), we get the symplectic group \( Sp_{2n} \) instead of \( O_{n, n} \). This proof works for the symplectic group, i.e. the homology stabilizes for \( n \geq 3k + 1 \). In this case the simplicial complex of flags of totally isotropic subspaces is the building associated to \( Sp_{2n} \), and since the Weyl group can be shown to be a finite euclidean reflection group, this building is homotopy equivalent to a wedge of spheres [3]. The proof in section 1 also shows that the building is a wedge of spheres. The stabilizer of the subspace \( <e_1, \ldots, e_p> \) has exactly the same form as in the \( O_{n, n} \) case; namely it consists of matrices of the form

\[
\begin{pmatrix}
\alpha & * & * \\
0 & A & * \\
0 & 0 & t_{\alpha-1}
\end{pmatrix}
\]
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where $\alpha \in \text{GL}_p(F)$ and $A \in \text{Sp}_{2(n-p)}$. The kernel of the projection

$$
\begin{pmatrix}
\alpha & * & * \\
0 & A & *
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & A & 0
\end{pmatrix}
$$

is all matrices of the form

$$
\begin{pmatrix}
I_p & a & b & c \\
0 & I_{n-p} & 0 & t_b \\
0 & 0 & I_{n-p} & -t_a \\
0 & 0 & 0 & I_p
\end{pmatrix},
$$

and the commutator subgroup $[N, N]$ is the matrices

$$
\begin{pmatrix}
I_p & 0 & 0 & x \\
0 & I_{n-p} & 0 & 0 \\
0 & 0 & I_{n-p} & 0 \\
0 & 0 & 0 & I_p
\end{pmatrix},
$$

where $x$ is any $p \times p$ matrix,

so $[N, N]$ = the additive abelian group of $p \times p$ matrices and $N/[N, N]$ = the additive abelian group of $p \times 2(n-p)$ matrices. The action of $Q^*$ on these groups is the same as in the $0_{n,n}$ case, and the rest of the proof is identical.

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REFERENCES


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