Rational Homology of Bianchi Groups*

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1. Introduction

Let $d$ be a square-free positive integer, and $O_{-d}$ the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Following work of Bianchi [3] and Humbert [10], Swan [18] described fundamental domains for the action of the groups $\text{SL}(2, O_{-d})$ on hyperbolic 3-space $\mathbb{H}$ and used these fundamental domains to give presentations for $\text{SL}(2, O_{-d})$ for some small values of $d$. However, the virtual cohomological dimension of $\text{SL}(2, O_{-d})$ is only 2, and there is in fact a 2-dimensional cellular retract of $\mathbb{H}$ which is invariant under the action of $\text{SL}(2, O_{-d})$ (see Ash [1]). In [12] Mendoza gave an explicit description of such a complex, and computed the cell structure for the cases where $O_{-d}$ is a Euclidean ring ($d = 1, 2, 3, 7, 11$). In [14] these complexes were used to compute the integral homology of $\text{SL}(2, O_{-d})$ and related groups in the Euclidean cases. The purpose of the present paper is to describe how to compute the cell structure and homology of Mendoza’s complexes for any $O_{-d}$; this program is carried out far enough to compute the rational homology of $\text{SL}(2, O_{-d})$ for values of $d$ such that the discriminant $D$ of $\mathbb{Q}(\sqrt{-d})$ is greater than $-100$. Of particular interest is the exact determination of the rank of the cuspidal cohomology $H^1_{\text{cusp}}$ for these values of $d$, corresponding to the dimension of the space of cuspidal harmonic automorphic forms for $\text{SL}(2, O_{-d})$ (see [16]). It is shown that for

$$d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}, \quad \dim H^1_{\text{cusp}} = 0.$$

Work of Zimmert [19], Baker [2], Grunewald and Schwermer [7], and Rohlf [13] implies that for all other values of $d$, the rank of the cuspidal cohomology is greater than zero; thus the above is a complete list of values of $d$ for which the cuspidal cohomology vanishes.

The computations in this paper also show that for discriminants greater than $-100$, all of the torsion in the integral homology of $\text{SL}(2, O_{-d})$ comes from the finite subgroups of $\text{SL}(2, O_{-d})$. Since Kramer [11] has completely determined the

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partially ordered set of finite subgroups for low discriminants, it is now possible to compute the integral homology of \( \text{SL}(2, O_{-d}) \) and related groups as in [14].

The paper is organized as follows. Section 2 briefly recalls well-known results about the action of \( \text{SL}(2, O_{-d}) \) on hyperbolic 3-space \( \mathbb{H} \), the quotient \( \mathbb{H}/\text{SL}(2, O_{-d}) \), and the rational and cuspidal cohomology of \( \text{SL}(2, O_{-d}) \). Section 3 describes Mendoza’s complex and lists the properties of this complex that we will need. Section 4 describes an algorithm for determining the cell structure of various pieces of the Mendoza complex and uses reduction theory to produce a computable finite bound on the number of steps required to complete the algorithm. Section 5 describes the group of symmetries of the complex, thereby simplifying the process of determining the cell structure. Section 6 contains theorems which aid in determining the quotient of the complex by \( \text{SL}(2, O_{-d}) \), and finally Sect. 7 contains the results of the computations for \( D > -100 \).

2. Preliminaries

The ring of integers \( O_{-d} \) is a \( \mathbb{Z} \)-lattice in the complex plane, generated by 1 and \( \omega \), where \( \omega = (1 + \sqrt{-d})/2 \) if \( d \) is congruent to 3 mod 4, and \( \omega = \sqrt{-d} \) otherwise. The discriminant \( D \) of \( \mathbb{Q}(\sqrt{-d}) \) is equal to \( -d \) if \( d \) is congruent to 3 mod 4, and \( -4d \) otherwise. Throughout this paper we assume \( D < -4 \) so that the only units in the ring \( O_{-d} \) are 1 and \( -1 \). For a complete discussion of the cases \( D = -3, -4 \), see [14].

We study the homology of \( \Gamma_d = \text{SL}(2, O_{-d}) \) by considering \( \Gamma_d \) as a discrete subgroup of \( \text{SL}(2, \mathbb{C}) \) and studying the action of \( \Gamma_d \) on the symmetric space \( \mathbb{H} = \text{SL}(2, \mathbb{C})/\text{SU}(2) \). \( \mathbb{H} \) is naturally identified with hyperbolic 3-space, and we will use the upper half-space model for \( \mathbb{H} \). Thus \( \mathbb{H} = \{(z, r) : z \in \mathbb{C} \text{ and } r \in \mathbb{R}, r > 0\} \), and the action of an element

\[
    g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

of \( \text{GL}(2, \mathbb{C}) \) on \( \mathbb{H} \) is given by the formula

\[
    g(z, r) = \frac{(az + b)(\overline{c}z + d) + acr^2}{(cz + d)(\overline{c}z + d) + c\overline{c}r^2}, \quad (cz + d)(\overline{c}z + d) + c\overline{c}r^2.
\]

This extends the standard action by linear fractional transformations of \( \text{GL}(2, \mathbb{C}) \) on the extended complex plane \( \mathbb{C} \cup \infty \) (thought of as the boundary of \( \mathbb{H} \)).

The points of \( \mathbb{Q}(\sqrt{-d}) \cup \infty \) in \( \mathbb{C} \cup \infty \) are called cusps. If \( \lambda \) is a cusp, we can write \( \lambda = \alpha/\beta \), with \( \alpha \) and \( \beta \) in \( O_{-d} \) (by convention, \( \infty = 1/0 \)). We can then associate to \( \lambda \) the element of the class group of \( O_{-d} \) represented by the ideal \( \langle \alpha, \beta \rangle \) of \( O_{-d} \) generated by \( \alpha \) and \( \beta \); this is called the class of \( \lambda \), and denoted \([\lambda] \). The class \([\lambda] \) is independent of the choice of \( \alpha \) and \( \beta \).

We now record some well-known facts about the action of \( \Gamma_d \) on \( \mathbb{H} \):

(2.2) The orbits of the action of \( \Gamma_d \) on the set of cusps are in one-to-one correspondence with the elements of the class group of \( O_{-d} \) [15].
(2.3) The orbit space \( M = H / I_d \) is the interior of a three-manifold with boundary; its boundary is the disjoint union of \( h \) toruses, where \( h \) is the class number of \( O_{-d} \) [6].

(2.4) \( H^*(I_d; \mathbb{Q}) \cong H^*(M; \mathbb{Q}). \)

Notation. Unless coefficients in homology and cohomology groups are given specifically, we will assume the coefficients are \( \mathbb{Q} \) with trivial action.

The contribution to the rational cohomology of \( I_d \) coming from the boundary of \( \infty \) is easily understood from the long exact sequence of the pair \((M, \partial M)\) and Poincaré duality. We have:

(2.1) Proposition. Let \( i : \partial M \to M \) be the inclusion map. Then \( \dim(\iota_* H_1(\partial M)) = h \) and \( \dim(\iota_* H_2(\partial M)) = h - 1 \), where \( h \) is the class number of \( O_{-d} \).

Proof. [15].

(2.6) Definition. The cuspidal cohomology \( H^*_\text{cusp} \) \((n = 1 \text{ or } 2)\) of \( I_d \) is the kernel of the map \( i^* : H^n(M) \to H^n(\partial M) \) induced by inclusion.

Although this is not the standard definition of cuspidal cohomology, it agrees with the standard definition in this case (see [7] for the correspondence).

(2.7) Lemma. There are short exact sequences

\[
H_1(\partial M) \xrightarrow{i_*} H_1(M) \to H^2_{\text{cusp}} \to 0
\]

and

\[
H_2(\partial M) \xrightarrow{i_*} H_2(M) \to H^1_{\text{cusp}} \to 0
\]

Proof. By Poincaré duality,

\[
H^1_{\text{cusp}} = \ker(\partial_* : H_2(M, \partial M) \to H_1(\partial M)).
\]

By the long exact homology sequence, this is

\[
= \cok(\iota_* : H_2(\partial M) \to H_2(M)).
\]

Similarly, \( H^2_{\text{cusp}} = \cok(\iota_* : H_1(\partial M) \to H_1(M)) \).

(2.8) Proposition. \( \dim(H^1_{\text{cusp}}) = \dim(H^2_{\text{cusp}}). \)

Proof. Since \( M \) is a three-manifold with boundary, we have the Euler characteristic \( \chi(M) = \chi(\partial M)/2. \) Since \( \partial M \) is a disjoint union of toruses, we have \( \chi(M) = 0 \), i.e.

\[
\dim(H_2(M)) - \dim(H_1(M)) + \dim(H_0(M)) = 0
\]

or

\[
\dim(H_2(M)) = \dim(H_1(M)) - 1.
\]

By (2.7), we have

\[
\dim(H_2(M)) = \dim(H^2_{\text{cusp}}) + \dim(\iota_*(H_1(\partial M))).
\]
and
\[ \dim(H_1(M)) = \dim(H_{\text{cusp}}^1) + \dim(i_\ast(H_2(\partial M))). \]

The proposition now follows from (2.5). □

Thus in order to compute the dimension of the rational homology of \( \SL(2, O_{-d}) \) we need only compute one of the homology groups \( H_1(M) \) or \( H_2(M) \). For most discriminants, we will actually determine the homotopy type of \( M \); however, in complicated cases it is easier to simply determine the homology groups. By (2.5) and (2.7), the only uncertainty lies in the dimension of the cuspidal group.

3. Mendoza’s Construction

In this section we briefly describe the construction of an \( \SL(2, O_{-d}) \)-invariant, two-dimensional cell complex which is a deformation retract of \( \H \). This construction is due to Mendoza. For details and proofs we refer to [12] or [14].

We first define the notion of the distance between a point \((z, r)\) of \( \H \) and a cusp \( \lambda \in \Q(\sqrt{-d}) \subset \partial \H \), due to Siegel [17]. Write \( \lambda = \alpha/\beta \), with \( \alpha \) and \( \beta \) in \( O_{-d} \). Then the distance from \((z, r)\) to \( \lambda \) is

\[ d((z, r), \lambda) = \frac{\|\beta z - \alpha\|^2 + \|\beta r\|^2}{r \cdot N\langle \alpha, \beta \rangle}, \tag{3.1} \]

where \( N\langle \alpha, \beta \rangle \) is the norm of the ideal of \( O_{-d} \) generated by \( \alpha \) and \( \beta \), and \( \| \| \) is the standard complex norm.

The set of points of \( \H \) which are equidistant from two cusps \( \lambda \) and \( \mu \), denoted \( S(\lambda, \mu) \), is a totally geodesic plane in \( \H \); in our model of \( \H \) this is a hemisphere or plane perpendicular to the boundary plane \( \C \). In particular, \( S(\infty, \alpha/\beta) \) is the hemisphere centered at \((\alpha/\beta, 0)\) with radius the square root of \( (N\langle \alpha, \beta \rangle)/N\langle \beta \rangle) \).

The following transformation rule tells us how the distance function behaves under the action of \( \GL(2, \Q(\sqrt{-d})) \).

(3.2) **Proposition.** If \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \GL(2, \Q(\sqrt{-d})) \) and \( \lambda = \alpha/\beta \) is a cusp, then

\[ d(g(z, r), g(\lambda)) = \frac{\|\det g \cdot N\langle \alpha, \beta \rangle\}}{N\langle g(\alpha, \beta) \rangle} d((z, r), \lambda), \]

where \( N\langle g(\alpha, \beta) \rangle \) is the ideal generated by \( ax + b\beta \) and \( cx + d\beta \).

**Proof.** The proof is straightforward using (2.1) and (3.1). □

As a corollary, we have that the distance function is invariant under elements of \( \SL(2, O_{-d}) \), i.e.

(3.3) **Proposition (Mendoza).** If \( g \in \SL(2, O_{-d}) \) then \( d(g(z, r), g(\lambda)) = d((z, r), \lambda) \).

**Proof.** Since \( g \in \SL(2, O_{-d}) \), it carries the lattice spanned by \( \alpha \) and \( \beta \) isomorphically onto the lattice spanned by \( ax + b\beta \) and \( cx + d\beta \); since \( \det g = 1 \), the areas of
fundamental domains for these lattices are the same, i.e. \( N(g(\alpha, \beta)) = N\langle g(\alpha, \beta) \rangle \).

(3.4) **Definition.** Let \( \lambda \) be a cusp. The minimal incidence set \( H(\lambda) \) of \( \lambda \) is the closure of the set of points in \( \mathbb{H} \) which are closer to \( \lambda \) than to any other cusp, i.e.

\[
H(\lambda) = \{ (z, r) : d((z, r), \lambda) \leq d((z, r), \mu) \text{ for all cusps } \mu \neq \lambda \}.
\]

(3.5) **Definition.** The Mendoza complex \( X_d \) is

\[
X_d = \bigcup_{\lambda \neq \mu} H(\lambda) \cap H(\mu) = \bigcup_{\lambda} \partial H(\lambda)
\]

It is instructive to visualize the case of \( \text{SL}(2, \mathbb{Z}) \) acting on the hyperbolic plane \( H \) (upper half space). Given a cusp \( p/q \in \mathbb{Q} \) with \( p \) and \( q \) relatively prime integers, the distance from a point \((x, r)\) of \( H \) to \((p/q, 0)\) in \( \partial H \) is equal to \((pq - p)^2 + (qr)^2)/r\); the set of points equidistant from \( p/q \) and \( \infty \) is the semicircle centered at \( p/q \) with radius \( 1/q \). \( H(\infty) \) consists of all points of \( H \) which lie above the set of semicircles with radius \( 1 \) centered at integer points \((p, 0)\). The complex \( X \) is the familiar tree for \( \text{SL}(2, \mathbb{Z}) \). Each minimal set \( H(p/q) \) is the closure of a connected component of \( H - X \).

We have the following facts about the complex \( X_d \):

(3.6) **Theorem** (Mendoza). (i) \( X_d \) is an \( \text{SL}(2, O_{-d}) \)-invariant, two-dimensional CW-complex, with cellular \( \text{SL}(2, O_{-d}) \)-action.

(ii) \( X_d \) is a deformation retract of \( \mathbb{H} \) by an \( \text{SL}(2, O_{-d}) \)-invariant deformation retraction.

(iii) \( X_d/\text{SL}(2, O_{-d}) \) is a finite CW-complex.

From this theorem it is evident that \( X_d/\text{SL}(2, O_{-d}) \) is a spine for the three-manifold \( M = \mathbb{H}/\text{SL}(2, O_{-d}) \). Thus to determine the homotopy type of \( \mathbb{H}/\text{SL}(2, O_{-d}) \), we need only determine the homotopy type of the finite complex \( X_d/\text{SL}(2, O_{-d}) \).

4. **Cell Structure**

In order to determine the cell structure of \( X_d = \bigcup_{\lambda} \partial H(\lambda) \), we first consider the simpler problem of determining the cell structure of \( H(\lambda) \) for a fixed cusp \( \lambda \). In order to accomplish this we will choose an element \( L \) of \( \text{GL}(2, \mathbb{Q}(\sqrt{-d})) \) which sends \( \infty \) to \( \lambda \), and then construct \( L^{-1}H(\lambda) \). This is equivalent to constructing the set of points closest to \( \infty \) for a few distance function, which depends on \( \lambda \).

Let \( n \) be the smallest positive real integer such that \( \lambda \in (1/n)O_{-d} \), and write \( \lambda = \alpha/n \) with \( \alpha \in O_{-d} \). If \( \lambda = \infty \), let \( \alpha = 1 \) and \( n = 0 \). The cusp \( \lambda \) and numbers \( \alpha \) and \( n \) will remain fixed throughout this section.

Fix \( L = \begin{bmatrix} \alpha & 1 \\ n & 0 \end{bmatrix} \in \text{GL}(2, \mathbb{Q}(\sqrt{-d})) \). If \( \lambda = \infty \), take \( L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{id.} \).
(4.1) Definition. The $\lambda$-distance from a point $(z, r)$ in $\mathbb{H}$ to a cusp $\mu$, $d_\lambda((z, r), \mu)$, is equal to $d(L(z, r), L(\mu))$.

Using this "new" distance, we can define the hemisphere $S_\lambda(\mu, v)$ to be the set of points equidistant from $\mu$ and $v$ with respect to $\lambda$-distance, and the minimal incidence set $H_\lambda(\mu)$ to be the closure of the set of points closer to $\mu$ than to any other cusp with respect to $\lambda$-distance.

Note that $H_\lambda(\infty)$ is the set of points in $\mathbb{H}$ which lie above all hemispheres $S_\lambda(\infty, \mu)$ for $\mu \in \mathbb{Q}(\sqrt{-d})$. The boundary $\partial H_\lambda(\infty)$ has a natural cell structure coming from the hemispheres $S_\lambda(\infty, \mu)$. To determine the cell determined by $S_\lambda(\infty, \mu)$, note that $S_\lambda(\infty, \mu) \cap S_\lambda(\infty, v)$ is either empty or is a semicircle perpendicular to the boundary plane. If the intersection is non-empty, let $l(\mu, v)$ be the line determined by the endpoints of this semicircle. Let $\pi : \mathbb{H} \to \mathbb{C}$ be the projection defined by $\pi(z, r) = z$. A point $(z, r)$ on $S_\lambda(\infty, \mu)$ is closer to $\infty$ than $v$ if and only if $z$ lies on the same side of $l(\mu, v)$ as $\mu$. Thus $\pi(S_\lambda(\infty, \mu) \cap H_\lambda(\infty))$ is the solution set of a collection of linear inequalities, determined by the hemispheres $S_\lambda(\infty, \mu)$ which intersect $S_\lambda(\infty, \mu)$ non-trivially.

This shows that $S_\lambda(\infty, \mu) \cap H_\lambda(\infty)$ is either empty or is a convex cell in $S_\lambda(\infty, \mu)$. In addition, if $S_\lambda(\infty, \mu) \cap H_\lambda(\infty)$ is a 0-cell or 1-cell, it lies in the boundary of a 2-cell $S_\lambda(\infty, v) \cap H_\lambda(\infty)$, where $v$ is a cusp such that $S_\lambda(\infty, \mu)$ intersects $S_\lambda(\infty, v)$ non-trivially.

Note also that $H_\lambda(\infty) = L^{-1}(H(\lambda))$. Since $L \in \text{GL}(2, \mathbb{C})$, $H_\lambda(\infty)$ is isometric to $H(\lambda)$; thus the cell structure of $\partial H(\lambda)$ is the same as the cell structure of $\partial(H_\lambda(\infty))$, which we will now determine.

(4.2) Proposition. Let $\mu = \gamma/\delta$ be a finite cusp, with $\gamma$ and $\delta$ in $O_{-d}$. Then $S_\lambda(\infty, \mu)$ is a hemisphere centered at $\mu$ with radius the square root of

$$\frac{N\langle L(\gamma, \delta) \rangle}{N\langle x, n \rangle N\langle \delta \rangle}.$$

Proof. By definition, we have

$$S_\lambda(\infty, \mu) = \{(z, r) : d_\lambda((z, r), \infty) = d_\lambda((z, r), \mu)\}$$

$$= \{(z, r) : d(L(z, r), L(\infty)) = d(L(z, r), L(\mu))\}.$$

By the transformation rule (3.2), this is

$$= \left\{(z, r) : \frac{\|\det L\|}{N\langle x, n \rangle} \cdot d((z, r), \infty) = \frac{\|\det L\|}{N\langle L(\gamma, \delta) \rangle} \cdot d((z, r), \mu)\right\}$$

$$= \left\{(z, r) : \frac{n}{N\langle x, n \rangle} \frac{1}{r} = \frac{n}{N\langle L(\gamma, \delta) \rangle} \cdot \frac{\|\delta z - \gamma\|^2 + \delta r^2}{r} \right\}$$

which simplifies to

$$= \left\{(z, r) : \|z - \mu\|^2 + r^2 = \frac{N\langle L(\gamma, \delta) \rangle}{N\langle x, n \rangle N\langle \delta \rangle} \right\}. \quad \Box$$

In order to actually compute $\partial H_\lambda(\infty)$, we need to know that its cell structure can be determined by a known finite number of calculations, i.e. that we can
determine which of the hemispheres $S_A(\infty, \mu)$ have non-empty intersection with $H_A(\infty)$ by considering a known finite collection of cusps $\mu$. The remainder of this section will establish this fact.

(4.3) **Lemma.** Let $\mu$ be a finite cusp. Write $\mu = \gamma/m$, with $m \in \mathbb{Z}$, $|m|$ minimal and $\gamma \in O_{-d}$. If $\lambda = \alpha/n$ is finite, then $N\langle L(\gamma, m) \rangle = N\langle \alpha \gamma + m, n \gamma \rangle \leq n^2m$. If $\lambda = \infty$, then $N\langle L(\gamma, m) \rangle \leq m$.

**Proof.** Assume $\lambda$ is finite. Note first that we can write $mn = n(\alpha \gamma + m) - \alpha(n \gamma)$, showing that $mn$ is in the ideal $\langle x\gamma + m, n \gamma \rangle$.

Let $\alpha = a_0 + a_1 \omega$ and $\gamma = c_0 + c_1 \omega$, with $a_0$, $a_1$, $c_0$, and $c_1$ in $\mathbb{Z}$. Since $|m|$ is minimal, we have $gcd(c_0, c_1, m) = 1$. Thus we may choose integers $x$, $y$, and $z$ with $xc_0 + yc_1 + zm = 1$.

If $d$ is not congruent to $3 \mod 4$, then $\omega = \sqrt{-d}$, and we have

$$nzw(\alpha \gamma + m)((x - a_0)\omega + (y + a_1) \omega) (n \gamma) = n(\omega + A),$$

where $A$ is some element of $\mathbb{Z}$. Thus $\langle x\gamma + m, n \gamma \rangle$ contains $n\langle \omega + A, m \rangle$ as a sublattice, and so $N\langle x\gamma + m, n \gamma \rangle$ divides $n^2N\langle \omega + a, m \rangle$, which divides $n^2m$.

The case $d$ congruent to $3 \mod 4$ is similar. The statement for $\lambda = \infty$ follows by the same methods. □

(4.4) **Definition.** An upper reduction constant for $\mathbb{Q}(\sqrt{-d})$ is a real number $C$, depending only on $d$, such that for any point $(z, r)$ in $\mathbb{H}$, there is at least one cusp $\mu$ with $d((z, r), \mu) \leq C$.

Note that if $C$ is an upper reduction constant for $\mathbb{Q}(\sqrt{-d})$ for the standard distance function, then it is also an upper reduction constant for $d_A$. We have the following result from classical reduction theory [8]:

(4.5) **Theorem.** Let $D$ be the discriminant of $\mathbb{Q}(\sqrt{-d})$. Then the square root of $-D/2$ is an upper reduction constant for $\mathbb{Q}(\sqrt{-d})$.

We can now prove the following useful proposition:

(4.6) **Proposition.** Let $\mu$ be a finite cusp, written $\mu = \gamma/m$ with $\gamma \in O_{-d}$, $m \in \mathbb{Z}$ and $|m|$ minimal. If $m > -N\langle \alpha, n \cdot (D/2) \rangle$, then $S_A(\infty, \mu) \cap H_A(\infty) = \emptyset$.

**Proof.** Assume $\lambda$ is finite. By Propositions (4.2) and (4.3), the square of the radius of $S_A(\infty, \mu)$ is equal to

$$\frac{N\langle \alpha \gamma + m, n \gamma \rangle}{N\langle \alpha, n \rangle \cdot m^2} \leq \frac{n^2}{N\langle \alpha, n \rangle \cdot m}.$$

Therefore, if $(z, r)$ is on $S_A(\infty, \mu)$, we have

$$d_A((z, r), \infty)^2 = \left( \frac{n}{N\langle \alpha, n \rangle} \right)^2 \frac{1}{r^2} \geq \frac{n^2}{N\langle \alpha, n \rangle^2} \frac{N\langle \alpha, n \rangle \cdot m}{n^2} \geq -\frac{D}{2}.$$
Thus if \( m > N \langle \alpha, \eta \rangle (-D/2) \), we have \( d_\lambda((z, r), \infty)^2 = m/N \langle \alpha, \eta \rangle > (-D/2) \). Since the square root of \(-D/2\) is an upper reduction constant for \( d_\lambda \), there is some cusp \( \nu \) with \( d_\lambda((z, r), \nu) < d_\lambda((z, r), \infty) \), i.e. \((z, r)\) is not in \( H_{\lambda}(\infty) \).

The case \( \lambda = \infty \) is similar. \( \square \)

As a consequence of this proposition, we know that we can determine \( \partial H_{\lambda}(\infty) \) by considering only cusps of the form \( \gamma/m \) with \( \gamma \in O_{-d}, \ m \in \mathbb{Z} \), and \( m \leq (-D \cdot N \langle \alpha, \eta \rangle)/2 \). The next step comes from noting that the set \( H(\lambda) \) is invariant under the stabilizer \( \Gamma(\lambda) \) of \( \lambda \) in \( \text{SL}(2, \mathbb{O}_{-d}) \); thus \( H(\infty) = L^{-1} H(\lambda) \) is invariant under the group \( L^{-1} \Gamma(\lambda)L \), which stabilizes \( \infty \) and hence acts as a group of translations of the boundary plane \( \mathbb{C} \). The following proposition determines a fundamental domain for this group of translations.

(4.7) Proposition. Let \( \lambda = \alpha/n \) be a finite cusp. Then the group \( L^{-1} \Gamma(\lambda)L \) acts on \( \partial \mathbb{H} - \{ \infty \} = \mathbb{C} \) by translation by elements of the fractional ideal \( n \langle \alpha, \eta \rangle^{-2} \).

Proof. Choose \( c, d \in \langle \alpha, \eta \rangle^{-1} \subset \mathbb{Q}(\sqrt{-d}) \) such that \( \alpha c + nd = 1 \), and let \( h \) be the matrix

\[
h = \begin{bmatrix} \alpha & c \\ n & d \end{bmatrix} \in \text{GL}(2, \mathbb{Q}(\sqrt{-d})).
\]

A straightforward computation shows that a matrix \( A \) is in \( \Gamma(\lambda) \) if and only if

\[
A = h \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \quad \text{for some} \quad x \in \langle \alpha, \eta \rangle^{-2}.
\]

Thus

\[
L^{-1} \Gamma(\lambda)L = \left\{ L^{-1} h \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} h^{-1} L : x \in \langle \alpha, \eta \rangle^{-2} \right\}
\]

\[
= \left\{ \begin{bmatrix} -n & -nx \\ 0 & -n \end{bmatrix} : x \in \langle \alpha, \eta \rangle^{-2} \right\}.
\]

The action of \( \begin{bmatrix} -n & -nx \\ 0 & -n \end{bmatrix} \) on \( \partial \mathbb{H} \) is the same as the action of \( \begin{bmatrix} 1 & nx \\ 0 & 1 \end{bmatrix} \), i.e. translation by \( nx \in n \langle \alpha, \eta \rangle^{-2} \). \( \square \)

(4.8) Corollary. The area of a fundamental domain for the action of \( L^{-1} \Gamma(\lambda)L \) on \( \mathbb{C} \) is

\[
\frac{n^2}{N \langle \alpha, \eta \rangle^2} \cdot \frac{\sqrt{-D}}{2} \quad (n \neq 0).
\]

Proof. The area of a fundamental domain for \( O_{-d} \) is \( \sqrt{-D}/2 \); the corollary follows from (4.7) because norm is multiplicative. \( \square \)
We collect our results so far in the following theorem:

(4.9) **Theorem.** Let $\lambda$ be a cusp and $L = \begin{bmatrix} x \\ n \\ 0 \end{bmatrix}$ the associated matrix. Let $P$ be a fundamental domain for the lattice $n \langle a, n \rangle^{-2}$. Then a fundamental domain for the action of the stabilizer $\Gamma(\lambda)$ of $\lambda$ on $\partial H(\lambda)$ is given by the union of the cells $H(\lambda) \cap H(L(\gamma/m))$, such that

(i) $m$ is a real integer, $0 < m < (-D, N \langle a, n \rangle)/2$

(ii) $\gamma$ is in $O_{-d}$ and $\gamma/m \in P$, and

(iii) $\dim [H(\lambda) \cap H(L(\gamma/m))] = 2$.

**Notation.** The cellular fundamental domain for $\Gamma(\lambda)$ given in the above theorem will be denoted $I(\lambda)$, and its image under $L^{-1}$ will be denoted $I_{\lambda}$ or $I_{\lambda}(\infty)$.

**Remarks.** For the purpose of finding all 2-cells in $\partial H(\lambda)$, the bound on $m$ given in the theorem is generally much too large. In the following example, we take $d = 23$, and compute $I_{\lambda}$ for $\lambda = \infty$, $(1 + \sqrt{-23})/4$, and $(1 - \sqrt{-23})/4$. The theorem implies that $S_2(\infty, \gamma/m)$ may intersect $H_2(\infty)$ only if $m \leq 11$. We will see that in fact $S_2(\infty, \gamma/m)$ intersects $H_2(\infty)$ in a 2-cell only for $m \leq 2$, and hence we can completely determine the cell structure of $I_{\lambda}$ by constructing the hemispheres $S_2(\infty, \gamma/m)$ for $m = 1$ and $m = 2$. What the theorem accomplishes is a guarantee that our calculations are complete if we have considered all cusps $\gamma/m$ with $m \leq 11$.

(4.10) **Example.** Let $d = 23$. The class group of $O_{-23}$ is $\mathbb{Z}/3$; the elements of the class group can be represented by the cusps $\infty$, $\omega/2$, and $(\omega + 1)/2$, where $\omega = (1 + \sqrt{-23})/2$. We compute $I_{\lambda}$ for each of these cusps.

**I. $\lambda = \infty$**

By Theorem (4.9), we need only determine $H(\infty) \cap H(\gamma/m)$ for $0 < m \leq 11$ and $\gamma = a + b \omega$ with $0 \leq a, b < m$.

We start with $m = 1$ and $\gamma = 0$. The cell $H(0) \cap H(\infty)$ is the portion of $S(0, \infty)$ which lies outside all hemispheres $S(\gamma/m, \infty)$ which intersect $S(0, \infty)$. After computing the intersections for $\gamma/m = -1/1$, $1/1$, $\omega/2$, $(\omega - 1)/2$, $-\omega/2$, and $(-\omega + 1)/2$, we have that $H(0) \cap H(\infty)$ is contained in the hexagon $\sigma$ on $S(0, \infty)$ with vertices

\[
p_1 = (z_1, r_1) = \left( \frac{4i}{\sqrt{23}}, \frac{\sqrt{7}/23}{\sqrt{23}} \right)
\]

\[
p_2 = (z_2, r_2) = \left( \frac{1 + 7i}{2}, \frac{\sqrt{5}/23}{\sqrt{23}} \right)
\]

\[
p_3 = (\tilde{z}_2, \tilde{r}_2)
\]

\[
p_4 = (-z_1, r_1)
\]

\[
p_5 = (-z_2, r_2)
\]

\[
p_6 = (-\tilde{z}_2, r_2).
\]
The points \((z, r)\) in \(\sigma\) with \(r\) smallest are the vertices \(p_2, p_3, p_5,\) and \(p_6,\) with \(1/5 < r^2 < 1/4.\) By Proposition (4.3), the square of the radius of \(S(\gamma/m, \infty)\) is less than \(1/m;\) thus \(S(\gamma/m, \infty)\) will not intersect \(\sigma\) if \(m \geq 5.\) We next check that \(S(\gamma/m, \infty) \cap \partial \sigma\) is empty or contained in \(\partial \sigma\) for all \(\gamma/m\) with \(m = 3\) or \(4;\) this shows that \(\sigma = H(0) \cap H(\infty).\)

We continue with \(m = 2;\) we must determine \(H(\infty) \cap H(\gamma/2)\) for \(\gamma = 1, \omega,\) and \(\omega + 1.\) The set \(H(\infty) \cap H(1/2)\) is contained in \(\sigma.\) We have \(H(\infty) \cap H(\omega/2)\) \(\subseteq S(\infty, \omega/2).\) By intersecting \(S(\infty, \omega/2)\) with \(S(\gamma/m, \infty)\) for \(\gamma/m = 0/1, \omega/1,\) \((\omega - 1)/2,\) and \((\omega + 1)/2,\) we obtain a parallelogram \(\tau\) on \(S(\infty, \omega/2)\) with vertices \(p_1, p_2, q_1 = (-z_1 + \omega, r_1)\) and \(q_2 = (-z_2 + \omega, r_2).\) As before, we need only check that no other hemispheres \(S(\gamma/m, \infty)\) with \(m \leq 4\) intersect the interior of \(\tau.\) \(H(\infty) \cap H((\omega + 1)/2)\) is another parallelogram \(\tau'.\) Figure 1 is the projection of \(\sigma \cup \tau \cup \tau'\) onto the complex plane:

![Fig. 1](image)

The translates of \(\sigma \cup \tau \cup \tau'\) by \(O_{-d} = \Gamma(\infty)\) cover the boundary plane, and we have shown that \(H(\infty) \cap H(\gamma/m)\) is either empty or is a cell in the union of these translates for all cusps \(\gamma/m.\) Thus \(\sigma \cup \tau \cup \tau'\) is a fundamental domain for the action of \(\Gamma(\infty)\) on \(\partial H(\infty);\) i.e. by definition, \(I(\infty) = \sigma \cup \tau \cup \tau'.\)

2. \(\lambda = \omega/2\)

The ideal \(2(\omega, 2)^{-1}\) is generated as a \(\mathbb{Z}\)-module by \(2\) and \((\omega + 1)/2.\) As before, we begin with \(m = 1;\) then \(\gamma\) may be \(0\) or \(1.\) We first compute \(H_\lambda(\infty) \cap H_\lambda(0);\) this lies on \(S_\lambda(0, \infty),\) which has radius \(1/\sqrt{2}.\) If we compute the intersections of \(S_\lambda(\gamma/m, \infty)\) with \(S_\lambda(0, \infty)\) for \(\gamma/m = 1/1,\) \(-1/1,\) \((\omega - 1)/2,\) and \((1 - \omega)/2,\) we obtain a parallelogram \(\tilde{\tau},\) whose lowest points \((z, r)\) have \(r^2 = 5/23.\) By Proposition (4.6), the square of the radius of \(S_\lambda(\gamma/m, \infty)\) is less than or equal to \(2/m,\) so we need only consider cusps \(\gamma/m\) with \(m \leq 9.\) We check that none of these spheres \(S_\lambda(\gamma/m, \infty)\) intersect the interior of \(\tilde{\tau}\) for such cusps \(\gamma/m,\) so \(\tilde{\tau} = H_\lambda(\infty) \cap H_\lambda(0).\) [The name \(\tilde{\tau}\) is deliberate; in fact \(H_\lambda(\infty) \cap H_\lambda(0) = L^{-1}(H(\infty) \cap H(\omega/2)),\) so we are just looking at our previous 2-cell \(\tau\) from the other side.]

To compute \(H_\lambda(\infty) \cap H_\lambda(1),\) we intersect \(S_\lambda(1, \infty)\) with the spheres \(S_\lambda(\gamma/m, \infty)\) with \(\gamma/m = 0/1, 2/1, (\omega - 1)/2, (\omega + 1)/2, (\omega - 3)/2, (3 - \omega)/2, (5 - \omega)/2\) to obtain an octagon \(\mathcal{Q}\) on \(S_\lambda(1, \infty).\) The lowest point \((z, r)\) on this octagon again has \(r^2 = 5/23,\) so we must check that no other hemispheres \(S_\lambda(\gamma/m, \infty)\) intersect the interior of \(\mathcal{Q}\) for \(m \leq 9.\)
In the process of this checking, we have shown that every hemisphere $S_\delta(\gamma/m, \infty)$ lies below the “roof” of translates of $\tilde{\tau}$ and $\tilde{\rho}$ by $L^{-1}T(\omega/2)L$; thus $I_{\omega/2} = \tilde{\rho} \cup \tilde{\tau}$ (Fig. 2).

3. $\lambda = (\omega + 1)/2$

The process is nearly identical to the process for $\lambda = \omega/2$, yielding Fig. 3. The labels $\tilde{\rho}$ and $\tilde{\tau}'$ are deliberate.

Note that the 2-cells in these pictures involve only cusps $\gamma/m$ with $m \leq 2$. We have in fact shown that $\sqrt{23/5}$ is the infimum of the set of upper reduction constants for $\mathbb{Q}(\sqrt{-23})$.

We will explain the apparent symmetries in these pictures in the next section.

5. Symmetries

In this section we describe the symmetries of the complex $X_\delta = \bigcup \partial H(\lambda)$, which will simplify our computations.

(5.1) **Definition.** $\partial H(\lambda)$ is isomorphic to $\partial H(\mu)$ if there is an orientation preserving isometry of $\mathbb{H}$ taking $H(\lambda)$ to $H(\mu)$.

Let $\lambda = \alpha/\beta$ be a cusp. Recall that the class of $\lambda$, denoted $[\lambda]$, is the element of the ideal class group of $O_{-\delta}$ represented by the ideal $\langle \alpha, \beta \rangle$ generated by $\alpha$ and $\beta$.

(5.2) **Proposition.** If $\lambda$ and $\mu$ are cusps with $[\lambda] = [\mu]$, then $\partial H(\lambda)$ is isomorphic to $\partial H(\mu)$.

**Proof.** By (2.1), we can find $g \in SL(2, O_{-\delta})$ with $g(\lambda) = \mu$. By (3.3), $g$ preserves the standard distance function, so sends $H(\lambda)$ isomorphically onto $H(\mu)$. □

(5.3) **Proposition.** Let $\lambda$ be a cusp. Then for any cusp $\mu$ and any $(z, r)$ in $\mathbb{H}$ we have

(i) $d((-z, r), -\mu) = d((z, r), \mu)$ and

(ii) $d_{\mathbb{H}}(z, r, \mu) = d_{\mathbb{H}}((z, r), \mu)$.

**Proof.** Part (i) is immediate from the definition of the distance function (3.1). Part (ii) follows from the definition of $\lambda$-distance (4.1), the transformation rule (3.2) and the fact that the norm of an ideal is the same as the norm of the conjugate ideal. □

(5.4) **Corollary.** $\partial H(\infty)$ is symmetric with respect to the origin and the plane $\text{Im}(z) = 0$.

(5.5) **Corollary.** $H_\delta(\infty)$ is the reflection of $H_\infty(\infty)$ through the plane $\text{Im}(z) = 0$. 
Note that (5.2)–(5.5) can be observed in the example \( d = 23 \) (Figs. 1–3 at the end of Sect. 4) if we take into account the fact that \( \lfloor (\omega - 1)/2 \rfloor = \lfloor \omega/2 \rfloor \).

There is one less obvious type of symmetry, given by the following theorem.

(5.6) **Theorem.** Let \( \lambda, \mu, \text{ and } v \) be cusps such that \( 2[\lambda] = 0 \) and \( [\mu] + [\lambda] = [v] \). Then \( \partial H(\mu) \) is isomorphic to \( \partial H(v) \).

**Proof.** Write \( \lambda = \alpha/\beta \) with \( \alpha, \beta \in O_{-d} \). Since \( 2[\lambda] = 0 \), there is an isomorphism
\[
g \colon O_{-d} \oplus O_{-d} \to \langle \alpha, \beta \rangle \oplus \langle \alpha, \beta \rangle,
\]
given by a matrix
\[
\begin{bmatrix}
x & y \\
z & w
\end{bmatrix}
\]
with \( \langle x, y \rangle = \langle z, w \rangle = \langle \alpha, \beta \rangle \) and \( \| \det g \| = N \langle \alpha, \beta \rangle \).

For any \( (s, t) \in O_{-d} \oplus O_{-d} \), we have
\[
g(s, t) = (xs + yt, zs + wt) \in \langle \alpha, \beta \rangle \langle s, t \rangle \oplus \langle \alpha, \beta \rangle \langle s, t \rangle.
\]
Since \( g \) is an isomorphism, \( xs + yt \) and \( zs + wt \) actually generate \( \langle \alpha, \beta \rangle \langle s, t \rangle \), i.e.
\[
(*) \quad \langle g(s, t) \rangle = \langle \alpha, \beta \rangle \langle s, t \rangle.
\]
Since norm is multiplicative, this gives
\[
(**) \quad N \langle g(s, t) \rangle = N \langle \alpha, \beta \rangle N \langle s, t \rangle = \| \det g \| \cdot N \langle s, t \rangle.
\]
Write \( \mu = \gamma/\delta \) with \( \gamma, \delta \in O_{-d} \). We can apply the transformation rule (3.2) to calculate
\[
d(g(z, r), g(\mu)) = \left( \frac{\| \det g \| \cdot N \langle \gamma, \delta \rangle}{N \langle g(\gamma, \delta) \rangle} \right) \cdot d((z, r), \mu)
= d((z, r), \mu) \text{ by } (**).
\]
Thus the map on \( \mathbb{H} \) induced by \( g \) preserves the distance function, so takes \( H(\mu) \) isomorphically to \( H(g(\mu)) \).

By (\(*\)), we have
\[
[g(\mu)] = [g(\gamma, \delta)] = [\langle \alpha, \beta \rangle \langle \gamma, \delta \rangle] = [\langle \alpha, \beta \rangle] + [\langle \gamma, \delta \rangle] = [\lambda] + [\mu] = [v].
\]
Therefore, by Proposition (5.2), we have \( \partial H(g(\mu)) \) isomorphic to \( \partial H(v) \). \( \square \)

The converse of this theorem is also true, i.e.:

(5.7) **Proposition.** If \( \partial H(\mu) \) is isomorphic to \( \partial H(v) \), then \( 2[\mu] = 2[v] \).

**Proof.** Any orientation-preserving isometry of \( \mathbb{H} \) is given by an element \( g \in (P) \text{SL}(2, \mathbb{C}) \). If \( gH(\mu) = H(v) \), then Proposition (3.7) implies that \( g^{-1} \Gamma(\mu)g = \Gamma(v) \), where \( \Gamma(x) \) denotes the stabilizer in \( \text{SL}(2, O_{-d}) \) of the cusp \( x \). As we noted in the proof of (3.7), this stabilizer \( \Gamma(x) \) is isomorphic to \( \langle a, b \rangle^{-2} \), for any
a, b ∈ O_{−d} with x = a/b. Since \([\langle a, b \rangle^{-2}] = −2[\langle a, b \rangle] = −2[x],\) we have \(Γ(μ)\) is isomorphic to \(Γ(ν)\) if and only if \(2[μ] = 2[ν].\)

Theorem (5.6) is of special interest when the subgroup \(C_2\) of elements of order two in the class group of \(O_{−d}\) is large. This group \(C_2\) is isomorphic to \((\mathbb{Z}/2)^{t−1}\), where \(t\) is the number of distinct prime divisors of the discriminant \(D\) of \(\mathbb{Q}(\sqrt{−d})\) (see, e.g. [9]). If \(t\) is large, the group of symmetries of \(X_d\) is large and we need to determine relatively few of the fundamental domains \(I_λ\) to obtain complete information on the cell structure of \(X_d\).

(5.8) Example. Let \(d = 5\). The class group of \(O_{−5}\) is \(\mathbb{Z}/2\); thus \(I(λ) ∪ I(∞)\) contains a fundamental domain for the action of \(\text{SL}(2, O_{−d})\) on \(X_d\), where \(λ\) is any non-principal cusp. The cell structure of \(I(λ)\) is identical to that of \(I(∞)\), since \([λ] + [λ] = [∞]\). Let \(g\) denote the isomorphism \(g : H(∞) → H(λ)\) as in Proposition (5.7). If two points \((z_1, r_1)\) and \((z_2, r_2)\) in \(H(∞)\) are equivalent by an element \(h \in \text{SL}(2, O_{−d})\), then \(ghg^{-1}\) identifies \(g(z_1, r_1)\) and \(g(z_2, r_2)\) in \(H(λ)\).

(5.9) Example. Let \(d = 87\). The class group of \(O_{−87}\) is cyclic of order 6, generated by the class \(a = [\sqrt[3]{ω}, 2]\). We can determine the cell structures of \(I(∞)\) and \(I(ω/2)\) by the methods of Sect. 4. The class \(3a\) is of order two in the class group; hence if \(λ\) is a cusp of this class, we have \(I(λ)\) isomorphic to \(I(∞)\). The class \(4a\) is \(a + 3a\); if \(μ\) is a cusp of class \(4a\), then \(I(μ)\) is isomorphic to \(I(ω/2)\). Since \(5a = (−a)\), and since conjugate ideals represent inverse elements of the class group, a cusp \(ν\) of class \(5a\) has \(I(ν)\) isomorphic to the reflection of \(I(ω/2)\). Finally, \(2a\) is the inverse of \(4a\), so the respective complexes are reflections of each other. Notice that in order to determine the structure of all complexes \(∂H(λ)\), we needed to determine \(I_λ\) for only two of the six classes of cusps.

6. Quotient

In Sect. 4 we found a cellular fundamental domain \(I(λ)\) for the action of the stabilizer \(Γ(λ)\) on \(∂H(λ)\). The quotient \(I(λ)/Γ(λ) = ∂H(λ)/Γ(λ)\) is a torus. In this section we find all additional identifications needed to determine the quotient \(X_d/\text{SL}(2, O_{−d}) = (∪ \partial H(λ))/\text{SL}(2, O_{−d}).\)

(6.1) Notation. Let \(λ_1, ..., λ_n\) be a set of cusps representing the set of ideal classes of \(O_{−d}\). We may assume \(λ_1 = ∞\) and \(λ_i = (ω−k_i)/n_i\) for \(i > 1\), where \(k_i ∈ \mathbb{Z}\) and \(n_i = N(ω−k_i, n_i)\) [5].

Let \(I = \bigcup_{i=1}^{n} I(λ_i)\). Since \(I(λ_i)\) is a fundamental domain for the action of \(Γ(λ_i)\) on \(∂H(λ_i)\), we have \(X_d/\text{SL}(2, O_{−d}) = I/\text{SL}(2, O_{−d}).\)

By well-known results in reduction theory [8], there are only a finite number of cusps within a given finite distance of a point in \(\mathbb{H}\). Thus we can make the following definitions:

(6.3) Definition. If \((z, r) ∈ \mathbb{H}\), the minimal cusp distance \(d(z, r)\) is equal to

\[
\min \{d((z, r), λ) : λ \text{ is a cusp}\}.
\]
(6.4) Definition. If \( \sigma \) is a cell of \( X_d \), the minimal cusp set \( \text{cusp}(\sigma) \) is the set of cusps \( \lambda \) such that \( d((z, r), \lambda) = d(z, r) \) for all points \((z, r)\) in the interior of \( \sigma \).

The following proposition is useful in determining which cells can be identified by elements of \( \text{SL}(2, O_{-d}) \):

(6.4) Proposition. Let \( \sigma \) be a cell of \( X_d \) and \( g \in \text{SL}(2, O_{-d}) \). Then cusp \( (g(\sigma)) \) is the set of cusps \( \lambda \) such that \( g^{-1}\lambda \in \text{cusp}(\sigma) \); i.e.

\[
\text{cusp}(g(\sigma)) = g(\text{cusp}(\sigma)).
\]

Proof. This follows immediately from the \( \text{SL}(2, O_{-d}) \)-invariance of the distance function [Proposition (3.3)]. \( \square \)

The next proposition gives a necessary condition for cells to be equivalent under \( \text{SL}(2, O_{-d}) \). See Sect. 4 for notation.

(6.5) Proposition. If \( \sigma \in I(\lambda_i) \) and \( \tau \in I(\lambda_j) \) are two 2-cells which are equivalent under \( \text{SL}(2, O_{-d}) \), then their images \( \sigma' \in I_{\mu} \) and \( \tau' \in I_{\mu} \) are congruent by an orientation-reversing Euclidean motion of upper half-space.

Proof. Since \( \sigma \) is a 2-cell of \( I(\lambda_i) \subset H(\lambda_i) \) we can write

\[
\sigma = H(\lambda_i) \cap H(\mu) \subset S(\lambda_i, \mu) \text{ for some cusp } \mu.
\]

Similarly,

\[
\tau = H(\lambda_j) \cap H(\nu) \subset S(\lambda_j, \nu) \text{ for some cusp } \nu.
\]

Choose an element \( g \) of \( \text{SL}(2, O_{-d}) \) with \( g(\sigma) = \tau \). By Proposition (6.4) we have

\[
\{g(\lambda_i), g(\mu)\} = \{\lambda_j, \nu\}.
\]

Claim 1. \( g(\lambda_i) = \nu \) and \( g(\mu) = \lambda_j \).

Proof. If \( i \neq j \), then since any element of \( \text{SL}(2, O_{-d}) \) preserves the class of a cusp, we have \([g(\lambda_i)] = [\lambda_i] + [\lambda_i] \); thus \( g(\lambda_i) = \nu \) and \( g(\mu) = \lambda_j \).

Now consider the case \( i = j \). Since both \( \sigma \) and \( \tau \) are cells of \( I(\lambda_i) \), and \( I(\lambda_i) \) is a fundamental domain for the action of \( \Gamma(\lambda_i) \) on \( \partial H(\lambda_i) \), \( g \) cannot stabilize \( \lambda_i \). Therefore in this case too, \( g(\lambda_i) = \nu \) and \( g(\mu) = \nu \). \( \square \)

Let

\[
L_i = \begin{bmatrix} \omega - k_i & 1 \\ n_i & 0 \end{bmatrix} \quad \text{and} \quad L_j = \begin{bmatrix} \omega - k_j & 1 \\ n_j & 0 \end{bmatrix}.
\]

Then \( \sigma' = L_i^{-1}(\sigma) \) and \( \tau' = L_j^{-1}(\tau) \). Let

\[
S_i = L_i^{-1}(S(\lambda_i, \mu)) = S_{\lambda_i}(\infty, L_i^{-1}(\mu))
\]

and

\[
S_j = L_j^{-1}(S(\lambda_j, \nu)) = S_{\lambda_j}(\infty, L_j^{-1}(\nu)).
\]

Note that \( L_j^{-1}gL_i(\sigma') = \tau' \), and in fact \( L_j^{-1}gL_i(S_i) = S_j \).

Claim 2. Let \( h = L_j^{-1}gL_i \). Then \( h|_{S_i} = \Phi|_{L_i} \) for some euclidean motion \( \Phi \) of upper half-space.
Proof. Since \( h \) is a hyperbolic isometry, it suffices to show that \( S_i \) and \( S_j = h(S_i) \) have the same radius.

Let \((z', r') \in S_i\) and \((w', s') = h(z', r') \in S_j\). Set \((z, r) = L_i(z', r')\), then
\[
1/r' = d((z', r'), \infty) = d(L_i^{-1}(z, r), L_i^{-1}(\lambda_i)).
\]
By the transformation rule, this is
\[
= (1/n_i) \cdot n_i \cdot d((z, r), \lambda_i)
= d((z, r), \lambda_i).
\]
Let \((w, s) = g(z, r)\); thus \((w', s') = L_j^{-1}(w, s)\). As above, we obtain
\[
1/s' = d((w, s), \lambda_j).
\]
Since distance is invariant under \( SL(2, O_{-d}) \), this is
\[
= d(g^{-1}(w, s), g^{-1}(\lambda_j))
= d((z, r), \mu) \text{ by Claim 1.}
\]
Since \((z, r) \in S(\lambda_i, \mu)\), this is
\[
= d((z, r), \lambda_i)
\]
which by our first calculation is
\[
= 1/r'.
\]
Thus \( s' = r' \) whenever \((w', s') = h|_{S_i}(z', r')\). This implies that the radius of \( S_i \) is equal to the radius of \( S_j \), as was to be shown. \( \square \)

It remains only to show that the Euclidean isometry \( \Phi \) which agrees with \( h \) on \( S_i \) must reverse orientation. Since \( \Phi \) preserves upper halfspace, this is equivalent to showing that \( \Phi|_{\mathbb{H}} \) reverses orientation. We know that \( h(\infty) = L_j(\infty) \) and \( h(L_j(\mu)) = \infty \) by Claim 1, i.e. \( h \) reverses the vertical direction. However, \( h \) preserves the orientation of \( \mathbb{H} \); therefore the projection of \( h|_{S_i} \) must reverse orientation on the boundary plane. \( \square \)

The next theorem tells us that in order to determine all identifications on \( I \) modulo \( SL(2, O_{-d}) \), we need only decide which 2-cells are identified. All other identifications are consequences of these and of identification by the stabilizers \( \Gamma(\lambda_i) \).

Note that for any \( \lambda \in \{\lambda_1, \ldots, \lambda_k\} \), the stabilizer of a 2-cell \( \sigma \) in \( I(\lambda) \) is either trivial or \( \mathbb{Z}/2 \). If the stabilizer is \( \mathbb{Z}/2 \), we can subdivide \( \sigma \) into two 2-cells, each with trivial stabilizer. We will call the resulting subdivided complex \( I'(\lambda) \). Let \( I' \) be the disjoint union of the \( I'(\lambda_i) \). Let \( p : I' \to X_d \) be the map induced by the inclusions \( I'(\lambda_i) \to X_d \), and \( \Psi \) the composition of \( p \) with the quotient map \( q : X_d \to X_d/SL(2, O_{-d}) \).

(6.6) **Theorem.** Let \( q \) and \( q' \) be i-cells \((i = 0 \text{ or } 1)\) of \( I' \) which are equivalent modulo \( SL(2, O_{-d}) \), i.e. \( \Psi(q) = \Psi(q') \). Then there is a sequence of pairs \((\sigma_k, q_k)\), \( 1 \leq k \leq n \), with \( q_1 = q \), \( q_n = q' \), \( \sigma_k \) a 2-cell of \( I' \) and \( q_k \subset \partial \sigma_k \) such that either
(i) \( q_{k+1} = g(q_k) \) for an element \( g \) in the stabilizer of some \( \lambda_i \) (\( g \) may be the identity) or
(ii) \( (\sigma_k, q_k) \) is equivalent to \( (\sigma_{k+1}, q_{k+1}) \) modulo \( SL(2, O_{-d}) \).

Proof. Let \( M \) denote the quotient \( \mathcal{H}/SL(2, O_{-d}) \), and \( M(k) \) the image in \( M \) of the \( k \)-skeleton of \( I' \), i.e. \( M(k) = \Psi(I'(k)) \).

Since the stabilizers of 2-cells in \( I' \) are trivial, the map \( \Psi \) gives a two-to-one correspondence between 2-cells of \( I' \) and 2-cells in \( M(2) \), which can be thought of as a one-to-one correspondence between 2-cells of \( I' \) and “sides” of 2-cells in \( M(2) \).

Let \( \bar{q} = \Psi(q) = \Psi(q') \subset M(i) \). Let \( U \) be a small neighborhood of \( \bar{q} \) in \( M \) which does not intersect any other \( i \)-cell of \( M(i) \), and let \( V = U - (U \cap M(2)) \).

Claim. The connected components of \( V \) are in one-to-one correspondence with the orbits of \( \Psi^{-1}(\bar{q}) \) under the stabilizers \( \Gamma(\lambda_i) \).

Proof. Let \( \tau \in \Psi^{-1}(\bar{q}) \). Then \( \tau \) is an \( i \)-cell of \( I(\lambda) \) for some \( \lambda \in \{ \lambda_1, \ldots, \lambda_k \} \). Let \( \gamma_\tau \) be the (unique) geodesic in \( \mathcal{H} \) from \( \lambda \) to the barycenter of \( \tau \). Then \( \Psi(\gamma_\tau) \) intersects \( U \) in exactly one component of \( V \). If \( \Psi(\gamma_\tau) \) enters the same connected component of \( V \) for some \( \tau' \neq \tau \) in \( \Psi^{-1}(\bar{q}) \), then the path \( \gamma_{\tau'}^{-1}\gamma_\tau \) is homotopic to a path which doesn’t intersect \( M(2) \), i.e. \( \gamma_{\tau'}^{-1}\gamma_\tau \) represents an element of the stabilizer \( \Gamma(\lambda) \) of \( \lambda \), and the \( i \)-cell \( \tau' \) is equivalent to \( \tau \) modulo \( \Gamma(\lambda) \).

To obtain the sequence \( (\sigma_k, q_k) \), connect the component of \( V \) corresponding to \( \bar{q} \) with the component corresponding to \( \bar{q}' \) by a path \( \alpha \) in \( U \) which intersects only 2-cells of \( M(2) \) (see Fig. 4). Then \( \sigma_k \) is the 2-cell corresponding to the \( k \)-th side of a 2-cell in \( M(2) \) encountered along \( \alpha \). If \( g_2 \) is an element of \( SL(2, O_{-d}) \) which identifies \( \sigma_{2k-1} \) with \( \sigma_{2k} \), then \( q_{2k} = g(q_{2k-1}) \). The 1-cell \( \partial q_{2k+1} \subset \partial \sigma_{2k+1} \) which maps to \( \bar{q} \) is equivalent to \( q_{2k} \) modulo a cusp stabilizer, since they correspond to the same component of \( V \). \( \square \)

\[ i = 1 \]

\[ {\bar{q}} \]

\[ V \cap M(2) \]

\[ U = V - (V \cap M(2)) = X(0, 1) \]

\[ \sigma_1 \]

\[ \sigma_2 \]

\[ \alpha \]

\[ \sigma_3 \]

\[ \sigma_4 \]

\[ q_4 \] is equivalent to \( q' \) modulo a cusp stabilizer.

Fig. 4
(6.7) Example. Let \( d = 23 \). In Example (3.10) we computed \( I_\lambda \) for \( \lambda = \infty, \omega/2, \) and \( (\omega + 1)/2 \). After translating \( I_\lambda \) back to \( I(\lambda) \subset X_d \), we have:

\[
I(\infty) = (H(\infty) \cap H(0)) \cup (H(\infty) \cap H(\omega/2)) \cup (H(\infty) \cap H((\omega + 1)/2))
= \sigma \cup \tau \cup \tau'
\]

\[
I(\omega/2) = (H(\infty) \cap H(\omega/2)) \cup (H(\omega + 1)/2) \cap H(\omega/2))
= \tau \cup \rho
\]

\[
I((\omega + 1)/2) = (H((\omega + 1)/2) \cap H(\omega/2)) \cup (H((\omega + 1)/2) \cap H(\infty))
= \tau' \cup \rho.
\]

by Proposition (6.4), the only possible identifications are by the stabilizers, and the identification of \( \sigma \) to itself given by the matrix \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \).

In this case, it is easiest to determine the quotient by simply drawing the pictures:

Figure 5 shows \( I(\omega/2)/\Gamma(\omega/2) \), where the shaded region is the image of \( \tau \) and the unshaded region is the image of \( \rho \):

![Fig. 5](image)

Figure 6 shows \( I((\omega + 1)/2)/\Gamma((\omega + 1)/2) \), where the shaded region is \( \tau' \) and the unshaded region is again \( \rho \):

![Fig. 6](image)

Figure 7 shows \( I(\infty)/\langle \Gamma(\infty), A \rangle \); the shaded region is \( \tau \cup \tau' \) and the unshaded region is the image of \( \sigma \). It is evident after staring for a moment that when glued together, the three pictures form a 2-complex homotopy equivalent to \( T^2 \vee S^2 \vee S^1 \), with the torus \( T^2 \) generated by \( \rho \), the 2-sphere \( S^2 \) by \( \tau \cup \tau' \) and the circle \( S^1 \) by \( \sigma \). In
particular, we have

$$\dim(H^i(\text{SL}(2, O_{-23}); \mathbb{Q}) = \begin{cases} 1 & i = 0 \\ 3 & i = 1 \\ 2 & i = 2 \end{cases}.$$ 

Note that $H^1_{\text{cusp}}(\text{SL}(2, O_{-23}) = 0.$

Fig. 7

(6.8) Remark. When $D$ is large, it is not always so easy to see what the quotient looks like. If we are only interested in computing the homology, it is easiest to compute $H_2$, using the boundary map from 2-cells to 1-cells. In this example ($d = 23$), the 2-cells and 1-cells in the quotient are identified as follows:

Fig. 8

The boundary matrix $\mathbb{Z}^4 \rightarrow \mathbb{Z}^8$ is:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 \\
0 & 1 & -1 & 1 \\
0 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$
After row-reducing this matrix, we see that its kernel has dimension 2, i.e. 
\( \dim(H_2 \text{SL}(2, O_{-23})) = 2 \).

7. Results

This section contains the results of the computations of homology of \( \text{SL}(2, O_{-d}) \) for discriminants \( D > -100 \). The quotient \( H/\text{SL}(2, O_{-d}) \) in every case computed has the homotopy type of a wedge product of circles \( S^1 \), 2-spheres \( S^2 \) and toruses \( T^2 \). The subcolumns under the heading “Homotopy Type” list the number of each of these which occurs in the wedge product.

(7.1) Table of results

<table>
<thead>
<tr>
<th>(-D)</th>
<th>(d)</th>
<th>Class group of ( O_{-d})</th>
<th>(\dim(H_{\text{cusp.}}))</th>
<th>Homotopy type of ( H/\text{SL}(2, O_{-d}))</th>
</tr>
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<tbody>
<tr>
<td>3</td>
<td>3</td>
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<td>0</td>
<td>(S^1) (S^2) (T^2)</td>
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(7.2) Example. Let $d=47$. The class group of $O_{-47}$ is $\mathbb{Z}/5 = \{1, a, a^2, a^3, a^4\}$, represented by the cusps \{\infty, \bar{\omega}/3, \omega/2, \bar{\omega}/2, \omega/3\} respectively. Figures 8–10 show computer-generated fundamental domains $I_\lambda$ for the action of the stabilizers of the cusps $\lambda = \infty, \omega/2, \omega/3$ on the boundary of the minimal sets $H_\lambda$. Each cell in $I_\lambda$ is equal to $H_\lambda \cap H_\mu$ for some cusp $\mu$, and the chart to the right of the figure identifies $\mu$ and the class $[\mu]$ for each cell. The labels on the 2-cells indicate which cells are identified modulo $\text{SL}(2, O_{-47})$. Note that the fundamental domains for the remaining cusp classes ($\lambda = 1/2$ and $\lambda = 1/3$) can be deduced from these pictures by the methods of Sect. 6.

\begin{table}[h]
\begin{tabular}{ccc}
2-cell & $\mu$ & $[\mu]$ \\
\hline
A & 0 & 1 \\
B & (\omega-1)/3 & a \\
C & $\omega/3$ & $a^4$ \\
D & (\omega-1)/2 & $a^3$ \\
E & $\omega/2$ & $a^2$ \\
F & (2\omega-2)/3 & a \\
G & 2\omega/3 & $a^4$
\end{tabular}
\end{table}

Fig. 9

\begin{table}[h]
\begin{tabular}{ccc}
2-cell & $\mu$ & $[\mu]$ \\
\hline
E & $\infty$ & 1 \\
H & (\omega-1)/2 & $a^3$ \\
J & 2\omega/3 & $a^4$
\end{tabular}
\end{table}

Fig. 10

\begin{table}[h]
\begin{tabular}{ccc}
2-cell & $\mu$ & $[\mu]$ \\
\hline
C & $\infty$ & 1 \\
K & (\omega+2)/3 & a \\
G & (\omega+1)/3 & 1 \\
J & $\omega/2$ & $a^2$ \\
L & (\omega-1)/3 & a
\end{tabular}
\end{table}

Fig. 11
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References

11. Kramer: Unpublished work

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