THE COMPLEX OF FREE FACTORS OF A
FREE GROUP

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1. Introduction

An important tool in the study of the group $GL(n, \mathbb{Z})$ is provided by the geometric realization of the partially ordered set (poset) of proper direct summands of $\mathbb{Z}^n$. The natural inclusion $\mathbb{Z}^n \to \mathbb{Q}^n$ gives a one-to-one correspondence between proper direct summands of $\mathbb{Z}^n$ and proper subspaces of $\mathbb{Q}^n$, so that this poset is isomorphic to the spherical building $X_n$ for $GL(n, \mathbb{Q})$. The term “spherical” comes from the Solomon-Tits theorem [8], which says that $X_n$ has the homotopy type of a bouquet of spheres:

Solomon-Tits Theorem. The geometric realization of the poset of proper subspaces of an $n$-dimensional vector space has the homotopy type of a bouquet of spheres of dimension $n - 2$.

The building $X_n$ encodes the structure of parabolic subgroups of $GL(n, \mathbb{Q})$: they are the stabilizers of simplices. $X_n$ also parametrizes the Borel-Serre boundary of the homogeneous space for $GL(n, \mathbb{R})$. The top-dimensional homology $H_{n-2}(X_n)$ is the Steinberg module $I_n$ for $GL(n, \mathbb{Q})$, and is a dualizing module for the homology of $GL(n, \mathbb{Z})$, i.e. for all coefficient modules $M$ there are isomorphisms

$$H^i(GL(n, \mathbb{Z}); M) \to H_{d-i}(GL(n, \mathbb{Z}); M \otimes I_n),$$

where $d = n(n - 1)/2$ is the virtual cohomological dimension of $GL(n, \mathbb{Z})$.

If one replaces $GL(n, \mathbb{Z})$ by the group $Aut(F_n)$ of automorphisms of the free group of rank $n$, the natural analog $FC_n$ of $X_n$ is the geometric realization of the poset of proper free factors of $F_n$. The abelianization map $F_n \to \mathbb{Z}^n$ induces a map from $FC_n$ to the poset of summands of $\mathbb{Z}^n$. In this paper we prove the analog of the Solomon-Tits theorem for $FC_n$:

Theorem 1.1. The geometric realization of the poset of proper free factors of $F_n$ has the homotopy type of a bouquet of spheres of dimension $n - 2$.

By analogy, we call the top homology $H_{n-2}(FC_n)$ the Steinberg module for $Aut(F_n)$. This leaves open some intriguing questions. It has recently been shown that $Aut(F_n)$ is a virtual duality group [1]; does the Steinberg module act as a

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dualizing module? There is an analog, called Autre space, of the homogeneous space for $GL(n, \mathbb{Z})$ and the Borel-Serre boundary; what is the relation between this and the “building” of free factors?

In [7], Quillen developed tools for studying the homotopy type of the geometric realization $|X|$ of a poset $X$. Given an order-preserving map $f : X \to Y$ (a “poset map”), there is a spectral sequence relating the homology of $|X|$, the homology of $|Y|$, and the homology of the “fibers” $|f/y|$, where

$$f/y = \{ x \in X / f(x) \leq y \}$$

with the induced poset structure.

To understand $FC_n$ then, one might try to apply Quillen’s theory using the poset map $FC_n \to X_n$. However, it seems to be difficult to understand the fibers of this map. Instead, we proceed by modelling the poset of free factors topologically, as the poset $B_n$ of simplices of a certain subcomplex of the “sphere complex” $S(M)$ studied in [2]. There is a natural poset map from $B_n$ to $FC_n$; we compute the homotopy type of $B_n$ and of the Quillen fibers of the poset map, and apply Quillen’s spectral sequence to obtain the result.

2. Sphere systems

Let $M$ be the compact 3-manifold obtained by taking a connected sum of $n$ copies of $S^1 \times S^2$ and removing a small open ball. A sphere system in $M$ is a non-empty finite set of disjointly embedded 2-spheres in the interior of $M$, no two of which are isotopic, and none of which bounds a ball or is isotopic to the boundary sphere of $M$. The complex $S(M)$ of sphere systems in $M$ is defined to be the simplicial complex whose $k$-simplices are isotopy class of sphere systems with $k + 1$ spheres.

Fix a basepoint $p$ on $\partial M$. The fundamental group $\pi_1(M, p)$ is isomorphic to $F_n$. Any automorphism of $F_n$ can be realized by a homeomorphism of $M$ fixing $\partial M$. A theorem of Laudenbach [4] implies that such a homeomorphism inducing the identity on $\pi_1(M, p)$ acts trivially on isotopy classes of sphere systems, so that in fact $Aut(F_n)$ acts on $S(M)$.

For $H$ a subset of $\pi_1(M, p)$, define $S_H$ to be the subcomplex of $S(M)$ consisting of isotopy classes of sphere systems $S$ such that $\pi_1(M - S, p) \supsetneq H$. Define $Y_H$ to be the subcomplex of $S_H$ consisting of isotopy classes of sphere systems $S$ such that $M - S$ is connected. For $H$ trivial, $S_H = S(M)$, and $Y_H$ is the complex $Y$ of [2].

The following theorems determine the homotopy types of $S_H$ and $Y_H$. The proofs rely upon the proofs for $S(M)$ and $Y$ given in [2].

THEOREM 2.1. $S_H$ is contractible.

Proof. We claim that the contraction of $S(M)$ in [2] restricts to a contraction of the subcomplex $S_H$. To show this, the key point is the following:
Lemma 2.2. Any two simplices in $S_H$ can be represented by sphere systems $\Sigma$ and $S$ such that every element of $H$ is representable by a loop disjoint from both $\Sigma$ and $S$.

Proof. Enlarge $\Sigma$ to a maximal sphere system $\Sigma'$, so the components of $M - \Sigma'$ are three-punctured spheres. By Proposition 1.1 of [2] we may isotope $S$ to be in normal form with respect to $\Sigma'$; this means that $S$ intersects each component of $M - \Sigma'$ in a collection of surfaces, each having at most one boundary circle on each of the three punctures.

We can represent a given element of $H$ by a loop $\gamma_0$ based at $p$, such that $\gamma_0$ is disjoint from $S$ and transverse to $\Sigma'$. The points of intersection of $\gamma_0$ with $\Sigma'$ divide $\gamma_0$ into a finite set of arcs, each entirely contained in one component of $M - \Sigma'$. Suppose one of these arcs $\alpha_0$, in a component $P$ of $M - \Sigma'$, has both endpoints on the same boundary sphere $\sigma$ of $P$. Since the map $\pi_0(\sigma - (S \cap \sigma)) \to \pi_0(P - (S \cap P))$ is injective (an easy consequence of normal form), there is an arc $\alpha'$ in $\sigma - (S \cap \sigma)$ with $\partial \alpha' = \partial \alpha$. Since $P$ is simply-connected, $\alpha$ is homotopic to $\alpha'$ fixing endpoints. This homotopy gives a homotopy of $\gamma_0$ eliminating the two points of $\partial \alpha'$ from $\gamma_0 \cap \Sigma'$, without introducing any intersection points with $S$. After repeating this operation a finite number of times, we may assume there are no remaining arcs of $\gamma_0 - (\gamma_0 \cap \Sigma')$ of the specified sort.

Now consider a homotopy $F : I \times I \to M$ of $\gamma_0$ to a loop $\gamma_1$ disjoint from $\Sigma$. Make $F$ transverse to $\Sigma'$ and look at $F^{-1}(\Sigma')$. This consists of a collection of disjoint arcs and circles. These do not meet the left and right edges of $I \times I$ since these edges map to the basepoint $p$.

We claim that every arc component of $F^{-1}(\Sigma')$ with one endpoint on $I \times \{0\}$ must have its other endpoint on $I \times \{1\}$. If not, choose an “edgmost” arc with both endpoints on $I \times \{0\}$, i.e. an arc such that the interval of $I \times \{0\}$ bounded by the endpoints contains no other point of $F^{-1}(\Sigma')$. Then $\gamma_0$ maps this interval to an arc $\alpha$ in $M - S$ which is entirely contained in one component $P$ of $M - \Sigma'$ and has both endpoints on the same boundary sphere of $P$, contradicting our assumption that all such arcs have been eliminated.

Since the loop $\gamma_1$ is disjoint from $\Sigma$, it follows that $\gamma_0$ must be disjoint from $\Sigma$, and by construction $\gamma_0$ was disjoint from $S$.

Since elements of $H$ are representable by loops disjoint from $\Sigma$ and $S$, these loops remain disjoint from sphere systems obtained by surgering $S$ along $\Sigma$, because such surgery produces spheres lying in a neighbourhood of $S \cup \Sigma$. This means that the contraction of $S(M)$ constructed in [2] restricts to a contraction of the subcomplex $S_H$. (Alternatively, we could use the simpler contraction technique of [3], which reverses the roles of $S$ and $\Sigma$.)

Definition. A simplicial complex $K$ is $k$-spherical if it is $k$-dimensional and $(k - 1)$-connected. By convention, a $(-1)$-spherical complex is a single point. A complex is spherical if it is $k$-spherical for some $k$. 

Now that we have \( S_H \) contractible, we use it to show that \( Y_H \) is spherical. In the course of the proof, we will need to consider complexes analogous to \( Y_H \) for manifolds with more than one boundary sphere. The next lemma shows that these have the same homotopy type as \( Y_H \).

Let \( M_k \) be the manifold obtained from the connected sum of \( n \) copies of \( S^1 \times S^2 \) by deleting \( k \) disjoint open balls rather than just a single ball. Choose the basepoint \( p \) on one of the spheres in \( \partial M \). For \( H \subseteq \pi_1(M_k, p) \), define \( Y_H(M_k) \) to be the complex of isotopy classes of sphere systems \( S \) in \( M_k \) with connected complement such that \( \pi_1(M - S, p) \supseteq H \).

**Lemma 2.3.** For \( k \geq 1 \), \( Y_H(M_k+1) \) deformation retracts onto a subcomplex isomorphic to \( Y_H(M_k) \).

**Proof.** Let \( \partial_0 \) and \( \partial_1 \) be two components of \( \partial M \), with \( p \in \partial_0 \). Fix a sphere \( \sigma \) in the interior of \( M_{k+1} \) which separates \( M_{k+1} \) into two components, one of which is a three-punctured sphere \( P \) containing \( \partial_0 \) and \( \partial_1 \), and the other of which is homeomorphic to \( M_k \).

Let \( S \) be a sphere system representing a simplex of \( Y_H(M_{k+1}) \). We may assume that \( S \) is in normal form with respect to \( \sigma \). This means that \( S \) intersects \( P \) in a collection of disjoint disks, each of which has boundary on \( \sigma \) and separates \( \partial_0 \) from \( \partial_1 \). Perform a sequence of modifications of \( S \) by transferring, one at a time, each disk of \( S \cap P \) to the other side of \( \partial_1 \). This has no effect on \( \pi_1(M - S, p) \), so the sphere systems which arise during this modification are still in \( Y_H(M_{k+1}) \). The final system resulting from transferring all the disks of \( S \cap P \) to the other side of \( \partial_1 \) can be isotoped to be disjoint from \( \sigma \).

This sequence of modifications can also be described as a sequence of surgeries on \( S \) using the disks in \( \sigma \) on the side away from \( \partial_0 \). As explained on pp. 48–49 of [2], such a surgery process defines a piecewise linear flow on the sphere complex. In the present case this flow gives a deformation retraction of \( Y_H(M_{k+1}) \) onto the subcomplex of sphere systems in \( M - \sigma \). This subcomplex can be identified with \( Y_H(M_k) \) by choosing the basepoint for \( M_k \) to be in \( \sigma \).

**Theorem 2.4.** Let \( H \) be a free factor of \( F_n = \pi_1(M, p) \). Then \( Y_H \) is \( (n - rk(H) - 1) \)-spherical.

**Proof.** Let \( i \leq n - rk(H) - 2 \). Any map \( g : S^i \to Y_H \) can be extended to a map \( \hat{g} : D^{i+1} \to S_H \) since \( S_H \) is contractible. We can assume \( \hat{g} \) is a simplicial map with respect to some triangulation of \( D^{i+1} \) compatible with its standard piecewise linear structure. We will redefine \( \hat{g} \) on the stars of certain simplices in the interior of \( D^{i+1} \) to make the image of \( \hat{g} \) lie in \( Y_H \).

To each sphere system \( S \) we associate a dual graph \( \Gamma(S) \), with one vertex for each component of \( M - S \) and one edge for each sphere in \( S \). The endpoints of the edge corresponding to \( s \in S \) are the vertices corresponding to the component or components adjacent to \( s \). We say a sphere system \( S \) is purely separating if \( \Gamma(S) \) has no edges which begin and end at the same vertex. Each sphere system \( S \)
has a purely separating core, consisting of those spheres in $S$ which correspond to the core of $\Gamma(S)$, i.e. the subgraph spanned by edges with distinct vertices. The purely separating core of $S \in S_H$ is empty if and only if $S$ is in $Y_H$.

Let $\sigma$ be a simplex of $D^{i+1}$ of maximal dimension among the simplices $\tau$ with $\hat{g}(\tau)$ purely separating. Note that all such simplices $\tau$ lie in the interior of $D^{i+1}$, since the boundary of $D^{i+1}$ maps to $Y_H$. Let $S = \hat{g}(\sigma)$, and let $N_0, \ldots, N_r (r \geq 1)$ be the connected components of $M - S$, with $p \in N_0$. A simplex $\tau$ in the link $lk(\sigma)$ maps to a system $T$ in the link of $S$, so that each $T_j = T \cap N_j$ is a sphere system in $N_j$ and $H \leq \pi_1(N_0 - T_0, p)$. Furthermore $N_j - T_j$ must be connected for all $j$ since otherwise the core of $\Gamma(S \cup T)$ would have more edges than $\Gamma(S)$, contradicting the maximality of $\sigma$. Thus $\hat{g}$ maps $lk(\sigma)$ into a subcomplex of $S_H$ which can be identified with $Y_H(N_0) \ast Y(N_1) \ast \cdots \ast Y(N_r)$.

Since $\sigma$ is a simplex in the interior of $D^{i+1}$, $lk(\sigma)$ is a sphere of dimension $i - \dim(\sigma)$. Each $N_j$ has fundamental group of rank $n_j < n$, so by Lemma 2.3 and induction, $Y_H(N_0)$ is $(n_0 - rk(H) - 1)$-spherical and, for $j \geq 1$, $Y(N_j)$ is $(n_j - 1)$-spherical. Hence $Y_H(N_0) \ast Y(N_1) \ast \cdots \ast Y(N_r)$ is spherical of dimension $(\sum_{j=0}^r n_j) - rk(H) - 1$.

Now $n = (\sum_{j} n_j) + rk(\pi_1(\Gamma(S))) = (\sum_{j} n_j) + m - r$ where $m$ is the number of spheres in $S$, i.e., edges in $\Gamma(S)$. Since a simplicial map cannot increase dimension, we have $\dim(\sigma) \geq m - 1$. Since $i \leq n - rk(H) - 2$, we have

$$i - \dim(\sigma) \leq n - rk(H) - 2 - \dim(\sigma)$$

$$\leq n - rk(H) - m - 1$$

$$= \left(\sum_{j} n_j\right) - rk(H) - 1 - r$$

$$< \left(\sum_{j} n_j\right) - rk(H) - 1.$$

Hence the map $\hat{g} : lk(\sigma) \rightarrow Y_H(N_0) \ast Y(N_1) \ast \cdots \ast Y(N_r)$ can be extended to a map of a disk $D^k$ into $Y_H(N_0) \ast Y(N_1) \ast \cdots \ast Y(N_r)$, where $k = i + 1 - \dim(\sigma)$. The system $S$ is compatible with every system in the image of $D^k$, so this map can be extended to a map $\sigma \ast D^k \rightarrow S_H$. We replace the star of $\sigma$ in $D^{i+1}$ by the disk $\partial(\sigma) \ast D^k$, and define $\hat{g}$ on $\partial(\sigma) \ast D^k$ using this map.

What have we improved? The new simplices in the disk $\partial(\sigma) \ast D^k$ are of the form $\sigma' \ast \tau$, where $\sigma'$ is a face of $\sigma$ and $\hat{g}(\tau) \subset Y_H(N_0)\ast Y(N_1)\ast \cdots \ast Y(N_r)$. The image of such a simplex $\sigma' \ast \tau$ is a system $S' \cup T$ such that in $\Gamma(S' \cup T)$ the edges corresponding to $T$ are all loops. Therefore any simplex in the disk $\partial(\sigma) \ast D^k$ with purely separating image must lie in the boundary of this disk, where we have not modified $\hat{g}$.

We continue this process, eliminating purely separating simplices until there is none in the image of $\hat{g}$. Since every system in $S_H - Y_H$ has a non-trivial purely separating core, in fact the whole disk maps into $Y_H$, and we are done.
3. Free factors

We now turn to the poset $FC_n$ of proper free factors of the free group $F_n$, partially ordered by inclusion. A $k$-simplex in the geometric realization $|FC_n|$ is a flag $H_0 < H_1 < \ldots < H_k$ of proper free factors of $F_n$, each properly included in the next. Each $H_i$ is also a free factor of $H_{i+1}$ (see [6, p. 117]), so that a maximal simplex of $|FC_n|$ has dimension $n - 2$.

We want to model free factors of $F_n$ by sphere systems in $Y = Y(M)$, by taking the fundamental group of the (connected) complement. A sphere system with $n$ spheres and connected complement, corresponding to an $(n - 1)$-dimensional simplex of $Y$, in fact has simply-connected complement. But we want to consider proper free factors, so instead we consider the $(n - 2)$-skeleton $Y^{(n-2)}$. Since $Y$ is $(n - 2)$-connected by Theorem 2.4, $Y^{(n-2)}$ is $(n - 2)$-spherical.

In order to relate $Y^{(n-2)}$ to $FC_n$, we take the barycentric subdivision $B_n$ of $Y^{(n-2)}$. Then $B_n$ is the geometric realization of a poset of isotopy classes of sphere systems, partially ordered by inclusion. If $S \subseteq S'$ are sphere systems, we have $\pi_1(M - S, p) \geq \pi_1(M - S', p)$, reversing the partial ordering. Taking fundamental group of the complement thus gives a poset map $f : B_n \to (FC_n)^{op}$, where $(FC_n)^{op}$ denotes $FC_n$ with the opposite partial ordering.

**Proposition 3.1.** $f : B_n \to (FC_n)^{op}$ is surjective.

**Proof.** Every simplex of $FC_n$ is contained in a simplex of dimension $n - 2$ so it suffices to show $f$ maps onto all $(n - 2)$-simplices. The group $\text{Aut}(F_n)$ acts transitively on $(n - 2)$-simplices of $FC_n$, and all elements of $\text{Aut}(F_n)$ are realized by homeomorphisms of $M$, so $f$ will be surjective if its image contains a single $(n - 2)$-simplex, which it obviously does.

**Corollary 3.2.** $FC_n$ is connected if $n \geq 3$.

**Proof.** Theorem 2.4 implies that $B_n$ is connected for $n \geq 3$. So, given any two vertices of $FC_n$, lift them to vertices of $B_n$ by Proposition 3.1, connect the lifted vertices by a path, then project the path back down to $FC_n$.

For any proper free factor $H$, let $B_{\geq H}$ denote the fiber $f/H$, consisting of isotopy classes of sphere systems $S$ in $B_n$ with $\pi_1(M - S, p) \geq H$.

**Proposition 3.3.** Let $H$ be a proper free factor of $F_n$. Then $B_{\geq H}$ is $(n - \text{rk}(H) - 1)$-spherical. If $\text{rk}(H) = n - 1$ then $B_{\geq H}$ is a single point.

**Proof.** The fiber $B_{\geq H}$ is the barycentric subdivision of $Y_H$, so is $(n - \text{rk}(H) - 1)$-spherical by Theorem 2.4.

Suppose $\text{rk}(H) = n - 1$, so that $\pi_1(M, p) = H \ast \langle x \rangle$ for some $x$. An element of $B_{\geq H}$ is represented by a sphere system containing exactly one sphere $s$, which is non-separating with $\pi_1(M - s, p) = H$. Suppose $s$ and $s'$ are two such spheres. Since $s$ and $s'$ are both non-separating, there is a homeomorphism $h$ of $M$ taking $s$
to \( s' \). Since automorphisms of \( H \) can be realized by homeomorphisms of \( M \) fixing \( s \), we may assume that the induced map on \( \pi_1(M, p) \) is the identity on \( H \).

**Claim.** The induced map \( h_\ast : \pi_1(M, p) \to \pi_1(M, p) \) must send \( x \) to an element of the form \( Ux^\pm V \), with \( U, V \in H \).

**Proof.** Let \( \{x_1, ..., x_{n-1}\} \) be a basis for \( H \), and let \( W \) be the reduced word representing \( h_\ast(x) \) in the basis \( \{x_1, ..., x_{n-1}, x\} \) for \( \pi_1(M, p) \). By looking at the map induced by \( h \) on homology, we see that the exponent sum of \( x \) in \( W \) must be \( \pm 1 \). Since \( h_\ast \) fixes \( H \), \( \{x_1, ..., x_{n-1}, W\} \) is a basis for \( \pi_1(M, p) \). If \( W \) contained both \( x \) and \( x^{-1} \), we could apply Nielsen automorphisms to the set \( \{x_1, ..., x_{n-1}, W\} \) until \( W \) was of the form \( x^\pm W_0x^\pm \). But \( \{x_1, ..., x_{n-1}, x^\pm W_0x^\pm \} \) is not a basis, since it is Nielsen reduced and not of the form \( \{x_1^\pm, ..., x_{n-1}^\pm, x^\pm\} \) (see [5], Prop. 2.8)

The automorphism fixing \( H \) and sending \( x \mapsto Ux^\pm V \) can be realized by a homeomorphism \( h' \) of \( M \) which takes \( s \) to itself (see [4], Lemma 4.3.1). The composition \( h'h^{-1} \) sends \( s' \) to \( s \) and induces the identity on \( \pi_1 \), hence acts trivially on the sphere complex. Thus \( s \) and \( s' \) are isotopic.

**Corollary 3.4.** \( FC_n \) is simply connected for \( n \geq 4 \).

**Proof.** Let \( e_0, e_1, ..., e_k \) be the edges of an edge-path loop in \( FC_n \), and choose lifts \( \tilde{e}_i \) of these edges to \( B_n \). Let \( e_{i-1}e_i \) or \( e_ie_0 \) be two adjacent edges of the path, meeting at the vertex \( H \). The lifts \( \tilde{e}_{i-1} \) and \( \tilde{e}_i \) may not be connected, i.e. \( \tilde{e}_{i-1} \) may terminate at a sphere system \( S' \) and \( \tilde{e}_i \) may begin at a different sphere system \( S \). However, both \( S \) and \( S' \) are in the fiber \( B_{\geq H} \), which is connected by Proposition 3.3, so we may connect \( S \) and \( S' \) by a path in \( B_{\geq H} \). Connecting the endpoints of each lifted edge in this way, we obtain a loop in \( B_n \), which may be filled in by a disk if \( n \geq 4 \), by Proposition 3.3. The projection of this loop to \( FC_n \) is homotopic to the original loop, since each extra edge-path segment we added projects to a loop in the star of some vertex \( H \), which is contractible. Therefore the projection of the disk kills our original loop in the fundamental group.

**Remark.** It is possible to describe the complex \( Y_H \) purely in terms of \( F_n \). Suppose first that \( H \) is trivial. Define a simplicial complex \( Z \) to have vertices the rank \( n-1 \) free factors of \( F_n \), with a set of \( k \) such factors spanning a simplex in \( Z \) if there is an automorphism of \( F_n \) taking them to the \( k \) factors obtained by deleting the standard basis elements \( x_1, ..., x_k \) of \( F_n \) one at a time. There is a simplicial map \( f : Y \to Z \) sending a system of \( k \) spheres to the set of \( k \) fundamental groups of the complements of these spheres. These fundamental groups are equivalent to the standard set of \( k \) rank \( n-1 \) factors under an automorphism of \( F_n \) since the homeomorphism group of \( M \) acts transitively on simplices of \( Y \) of a given dimension, and the standard \( k \) factors are the fundamental groups of the complements of the spheres in a standard system in \( Y \). The last statement of Proposition 3.3 says that \( f \) is a bijection on vertices, so \( f \) embeds \( Y \) as a subcomplex of \( Z \). The maximal simplices in \( Y \) and \( Z \) have dimension \( n-1 \) and the groups \( Homeo(M) \)
and $\text{Aut}(F_n)$ act transitively on these simplices, so $f$ must be surjective, hence an isomorphism. When $H$ is nontrivial, $f$ restricts to an isomorphism from $Y_H$ onto the subcomplex $Z_H$ spanned by the vertices which are free factors containing $H$.

We are now ready to apply Quillen's spectral sequence to compute the homology of $B_n$ and thus prove the main theorem.

**Theorem 3.5.** $FC_n$ is $(n - 2)$-spherical.

**Proof.** We prove the theorem by induction on $n$. If $n \leq 4$, the theorem follows from Corollaries 3.2 and 3.4.

Quillen's spectral sequence [7, 7.7] applied to $f : B_n \to (FC_n)^{op}$ becomes

$$E^2_{p,q} = H_p(FC_n; H \mapsto H_q(B_{\geq H})) \Rightarrow H_{p+q}(B_n),$$

where the $E^2$-term is computed using homology with coefficients in the functor $H \mapsto H_q(B_{\geq H})$.

For $q = 0$, Corollary 3.2 gives $H_0(B_{\geq H}) = \mathbb{Z}$ for all $H$, so $E^2_{p,0} = H_p(FC_n, \mathbb{Z})$.

For $q > 0$, we have $E^2_{p,q} = H_p(FC_n; H \mapsto \tilde{H}_q(B_{\geq H}))$, and we follow Quillen ([7], proof of Theorem 9.1) to compute this.

For a subposet $A$ of $FC_n$, let $L_A$ denote the functor sending $H$ to a fixed abelian group $L$ if $H \in A$ and to 0 otherwise. Set $U = FC_{\leq H} = \{H' \in FC_n | H' \leq H\}$ and $V = FC_{< H} = \{H' \in FC_n | H' < H\}$. Then

$$H_i(FC_n; L_V) = H_i(V; L),$$

and

$$H_i(FC_n; L_U) = H_i(U; L) = \begin{cases} L & \text{if } i = 0 \text{ and } \\ 0 & \text{otherwise,} \end{cases}$$

since $|U|$ is contractible. The short exact sequence of functors

$$1 \to L_V \to L_U \to L_{\{H\}} \to 1$$

gives a long exact homology sequence, from which we compute

$$H_i(FC_n; L_{\{H\}}) = \tilde{H}_{i-1}(FC_{< H}; L).$$

Now, $H \mapsto \tilde{H}_q(B_{\geq H})$ is equal to the functor

$$\bigoplus_{rk(H) = n-q-1} \tilde{H}_q(B_{\geq H})_{\{H\}}$$

since $B_{\geq H}$ is $(n - rk(H) - 1)$-spherical by Proposition 3.3. Thus

$$E^2_{p,q} = H_p(FC_n; H \mapsto \tilde{H}_q(B_{\geq H})) = \bigoplus_{rk(H) = n-q-1} H_p(FC_n; \tilde{H}_q(B_{\geq H})_{\{H\}})$$

$$= \bigoplus_{rk(H) = n-q-1} \tilde{H}_{p-1}(FC_{< H}; \tilde{H}_q(B_{\geq H})).$$
Free factors of $F_n$ contained in $H$ are also free factors of $H$. Since $H$ has rank $< n$, $FC_{< H}$ is $(rk(H) - 2)$-spherical by induction. Therefore $E^2_{p,q} = 0$ unless $p - 1 = (n - q - 1) - 2$, i.e. $p + q = n - 2$. Since all terms in the $E^2$-term of the spectral sequence are zero except the bottom row for $p \leq n - 2$ and the diagonal $p + q = n - 2$, all differentials are zero and we have $E^2 = E^\infty$ as in the following diagram:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & \\
E^2_{0,n-2} & 0 & 0 & 0 & 0 & \\
0 & E^2_{1,n-3} & 0 & 0 & 0 & \\
0 & 0 & E^2_{2,n-4} & 0 & 0 & \\
0 & 0 & 0 & E^2_{3,n-5} & 0 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
H_0(FC_n) & H_1(FC_n) & H_2(FC_n) & H_3(FC_n) & \ldots & H_{n-2}(FC_n) & 0
\end{array}
\]

Since $FC_n$ is connected and the spectral sequence converges to $H_*(B_n)$, which is $(n - 3)$-connected, we must have $\tilde{H}_i(FC_n) = 0$ for $i \neq n - 2$. Since $FC_n$ is simply-connected by Corollary 3.4, this implies that $FC_n$ is $(n - 3)$-connected by the Hurewicz theorem. The theorem follows since $FC_n$ is $(n - 2)$-dimensional.

4. The Cohen-Macaulay Property

In a PL triangulation of an $n$-dimensional sphere, the link of every $k$-simplex is an $(n - k - 1)$-sphere. A poset is said to be Cohen-Macaulay of dimension $n$ if its geometric realization is $n$-spherical and the link of every $k$-simplex is $(n - k - 1)$-spherical (see [7]). Spherical buildings are Cohen-Macaulay, and we remark that $FC_n$ also has this nice local property.

To see this, let $\sigma = H_0 < H_1 < \ldots < H_k$ be a $k$-simplex of $FC_n$. The link of $\sigma$ is the join of subcomplexes $FC_{H_i,H_{i+1}}$ of $FC_n$ spanned by free factors $H$ with $H_i < H < H_{i+1}$ ($-1 \leq i \leq k$, with the conventions $H_{-1} = 1$ and $H_{k+1} = F_n$). Counting dimensions, we see that it suffices to show that for each $r$ and $s$ with $0 \leq s < r$, the poset $FC_{r,s}$ of proper free factors $H$ of $F_r$ which properly contain $F_s$ is $(r - s - 2)$-spherical. The proof of this is identical to the proof that $FC_n$ is $(n - 2)$-spherical, after setting $n = r$ and replacing the complex $Y$ by $Y_{F_r}$.

5. The map to the building

As mentioned in the introduction, the abelianization map $F_n \to \mathbb{Z}^n$ induces a map from the free factor complex $FC_n$ to the building $X_n$, since summands of $\mathbb{Z}^n$ correspond to subspaces of $Q^n$. Since the map $Aut(F_n) \to GL(n, \mathbb{Z})$ is surjective, every basis for $\mathbb{Z}^n$ lifts to a basis for $F_n$, and hence every flag of summands of $\mathbb{Z}^n$ lifts to a flag of free factors of $F_n$, i.e. the map $FC_n \to X_n$ is surjective.

Given a basis $\{v_1, \ldots, v_n\}$ for $Q^n$, consider the subcomplex of $X_n$ consisting of
all flags of subspaces of the form \( \langle v_{i_1} \rangle \subset \langle v_{i_1}, v_{i_2} \rangle \subset \ldots \subset \langle v_{i_1}, \ldots, v_{i_n} \rangle \). This subcomplex can be identified with the barycentric subdivision of the boundary of an \((n - 1)\)-dimensional simplex, so forms an \((n - 2)\)-dimensional sphere in \(X_n\), called an apartment.

The construction above applied to a basis of \( F_n \) instead of \( \mathbb{Q}^n \) yields an \((n - 2)\)-dimensional sphere in \( FC_n \). In particular, \( H_{n-2}(FC_n) \) is non-trivial. This sphere maps to an apartment in \( X_n \) showing that the induced map \( H_{n-2}(FC_n) \to H_{n-2}(X_n) \) is also non-trivial.

The property of buildings which is missing in \( FC_n \) is that given any two maximal simplices there is an apartment which contains both of them. For example, for \( n = 3 \) and \( F_3 \) free on \( \{x, y, z\} \) there is no "apartment" which contains both the one-cells corresponding to \( \langle x \rangle \subset \langle x, y \rangle \) and \( \langle yxy^{-1} \rangle \subset \langle x, y \rangle \), since \( x \) and \( yxy^{-1} \) do not form part of a basis of \( F_3 \).

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