

AUTOMORPHISMS OF FREE GROUPS AND OUTER SPACE

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ABSTRACT: This is a survey of recent results in the theory of automorphism groups of finitely-generated free groups, concentrating on results obtained by studying actions of these groups on Outer space and its variations.

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§0. Introduction

1. History and motivation

This paper is a survey of recent results in the theory of automorphism groups of finitely-generated free groups, concentrating mainly on results which have been obtained by studying actions on a certain geometric object known as Outer space and its variations.

The study of automorphism groups of free groups in itself is decidedly not new; these groups are basic objects in the field of combinatorial group theory, and have been studied since the very beginnings of the subject. Fundamental contributions were made by Jakob Nielsen starting in 1915 and by J.H.C. Whitehead in the 1930's to 1950's. In their 1966 book, Chandler and Magnus remark that these groups “have attracted a tremendous amount of research work in spite of gaps of decades in the sequences of papers that deal with them” [27]. And, in spite of the work that was done, Roger Lyndon remarked in 1977 that there was still much more *unknown* about these groups than known, and he included the following in his list of major unsolved problems in group theory:

“Determine the structure of $\text{Aut } F$, of its subgroups, especially its finite subgroups, and its quotient groups, as well as the structure of individual automorphisms.” [88]

Since Lyndon made his list, geometric methods introduced by work of Thurston and Gromov have swayed the focus of “combinatorial group theory” to that of “geometric group theory.” In order to study a group, one now looks for a nice space on which the group acts, uses topological and geometric methods to study the space, and translates the results back into algebraic information about the group. Borel and Serre had used geometric methods to study arithmetic and S -arithmetic groups via their action on homogeneous spaces and buildings. Thurston studied the mapping class group of a surface via the dynamics of its action on the

Teichmüller space of the surface. Gromov revolutionized group theory by considering *all* finitely-generated groups as metric spaces, and studying the groups via the geometry of their actions on themselves.

Arithmetic groups and mapping class groups are especially compelling models for the study of automorphism groups of free groups. The group $Out(F_2)$ of outer automorphisms of the free group of rank 2 is both arithmetic (isomorphic to $GL(2, \mathbf{Z})$) and a mapping class group (isomorphic to the mapping class group of a torus or a once-punctured torus). In general, the abelianization map $F_n \rightarrow \mathbf{Z}^n$ induces a map from $Aut(F_n)$ to $GL(n, \mathbf{Z})$ which is trivial on inner automorphisms so factors through $Out(F_n)$. Nielsen [100] showed that this map is surjective, and that for $n = 2$ it is an isomorphism; Magnus [90] showed that the kernel is always finitely generated, and Baumslag and Taylor showed that the kernel is torsion-free [6]. The connection with mapping class groups comes via the fact that if S is a punctured surface with fundamental group F_n , then the subgroup of $Out(F_n)$ which stabilizes the cyclic words represented by the boundary loops of the surface is isomorphic to the mapping class group of the surface (see Magnus [91] for some small values of n , Zieschang [131] in general).

New geometric methods motivated by techniques which were successful for arithmetic and mapping class groups have enabled us to make a great deal of progress in understanding automorphism groups of free groups. Culler and Vogtmann constructed a contractible “Outer space” on which the group $Out(F_n)$ acts properly; the idea is to mimic the construction of Teichmüller space using metric graphs in place of Riemann surfaces. Bestvina and Handel introduced “train track” techniques inspired by Thurston’s theory of train tracks for surfaces; these model a single automorphism of a free group by a particularly nice homotopy equivalence of a graph. Bestvina, Handel and Feighn, as well as Lustig, introduced spaces of “laminations” on a free group, as an analog of the boundary of Teichmüller space in Thurston’s theory of surface automorphisms. Bestvina and Paulin considered the sequence of actions of F_n on its Cayley graph given by applying powers of an automorphism of F_n , and showed how to take a “Gromov-Hausdorff limit” of these actions. Using these basic geometric tools, we can now show that $Out(F_n)$ has strong algebraic finiteness properties, we can compute algebraic invariants such as homology and Euler characteristic, our knowledge of the subgroup structure of $Out(F_n)$ is greatly expanded, we can analyze the structure of a single automorphism quite precisely, and we can attack algorithmic questions such as solvability of the conjugacy problem.

2. Where to get basic facts

Two standard references for classical results on automorphisms of free groups are the 1966 book *Combinatorial Group Theory*, by Magnus, Karass and Solitar [91] and the 1977 book by Lyndon and Schupp of the same title [89].

Much of the work described in this survey is based on methods invented by J.H.C. Whitehead (see [130]) and by John Stallings and Steve Gersten (see [118], [119], [120], [121] [122], [123], [52], [53], [54]). In particular, the “star

graph” described by Whitehead and the process of “folding” due to Stallings play essential roles.

3. What’s in this paper

The paper is divided into four main sections. The first describes Outer space, its boundary and its “spine” and some variations and other complexes on which groups of automorphisms of a free group act. The second lists algebraic results and tries to give some indication of how geometry and topology are used in obtaining these results. The third part is a collection of some open questions, and the fourth part is a list of references to papers in the subject.

4. What’s not in this paper

This paper began life as a file in which I wrote down recent results about automorphisms of free groups for my own reference. Since I know what’s in my own papers on the subject, it contains a fairly complete list of results from those papers. I do not pretend to any claims of completeness for the work of other people; there is certainly much more written on the subject than is explicitly mentioned here. Many facts are mentioned with little attempt at motivation or indication of proof. I hope that the list of references at the end of the paper will be useful to people interested both in learning more details and in finding out about other results in the field. Just a few of the important topics which are not discussed in detail are spaces of “laminations” of a free group, explicit descriptions of various normal forms for an automorphism, the construction of groups with interesting properties using “mapping tori” of automorphisms of a free group, and properties of the subgroup of pure symmetric automorphisms of a free group.

§1. Spaces and complexes

1. Outer space

In [38] Culler and Vogtmann introduced a space X_n on which the group $Out(F_n)$ acts with finite point stabilizers, and proved that X_n is contractible. Peter Shalen later invented the name “Outer space” for X_n . Outer space with the action of $Out(F_n)$ can be thought of as analogous to a homogeneous space with the action of an arithmetic group, or to the Teichmüller space of a surface with the action of the mapping class group of the surface.

The basic idea of Outer space is that points correspond to graphs with fundamental group isomorphic to F_n , and that $Out(F_n)$ acts by changing the isomorphism with F_n . Each graph comes with a metric, and one moves around the space by varying the lengths of edges of a graph. Edge lengths are allowed to become zero as long as the fundamental group is not changed. When an edge length becomes zero, there are several ways to resolve the resulting vertex to obtain a new nearby graph; in terms of the space, this phenomenon results in the fact that, unlike homogeneous spaces and Teichmüller spaces, Outer space is not a manifold.

We now define Outer space formally. Fix a graph R_n with one vertex and n edges, and identify the free group $F_n = F\langle x_1, \dots, x_n \rangle$ with $\pi_1(R_n)$ in such a way that each generator x_i corresponds to a single oriented edge of R_n . Under this identification, each reduced word in F_n corresponds to a reduced edge-path loop starting at the basepoint of R_n . An automorphism $\phi: F_n \rightarrow F_n$ is represented by the homotopy equivalence of R_n , which sends the (oriented) edge loop labelled x_i to the edge-path loop labelled $\phi(x_i)$.

Points in Outer space are defined to be equivalence classes of *marked metric graphs* (g, Γ) , where

- Γ is a graph with all vertices of valence at least three;
- $g: R_n \rightarrow \Gamma$ is a homotopy equivalence, called the *marking*;
- each edge of Γ is assigned a positive real length, making Γ into a metric space via the path metric;

The equivalence relation is given by $(g, \Gamma) \sim (g', \Gamma')$ if there is a homothety $h: \Gamma \rightarrow \Gamma'$ with $g \circ h$ homotopic to g' . (Recall h is a *homothety* if there is a constant $\lambda > 0$ with $d(h(x), h(y)) = \lambda d(x, y)$ for all $x, y \in \Gamma$.)

We can represent a point (g, Γ) in Outer space by drawing the graph Γ , choosing a maximal tree T , and labelling all edges which are not in T (see Figure 1). Each edge label consists of an orientation and an element of F_n . The labels determine a map $h: \Gamma \rightarrow R_n$ which sends all of T to the basepoint of R_n and sends each edge in $\Gamma - T$ to the edge-path loop in R_n indicated by the label; we choose the labels so that this map h is a homotopy inverse for g . (Note that h is a homotopy equivalence if and only if the set of words labelling edges forms a basis for F_n .) These representations are useful even though they are clearly not unique; they depend on the choice of maximal tree T and the choice of the labels.

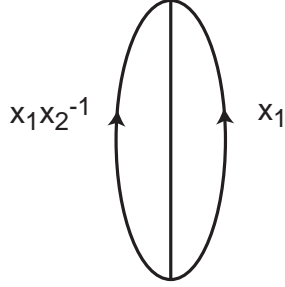


Figure 1: Representation of a point in Outer space

In order to describe the topology on Outer space, we consider the set \mathcal{C} of conjugacy classes, or cyclic words, in F_n . We define a map from Outer space X_n to the infinite-dimensional projective space $\mathbf{RP}^{\mathcal{C}}$ as follows. Given a metric marked graph (Γ, g) , we assign to each cyclically reduced word w the length of the unique cyclically reduced edge path loop in Γ homotopic to $g(w)$. This map is injective, by [37], and we give X_n the subspace topology.

In this topology, X_n decomposes into a disjoint union of open simplices, as follows. We first normalize by assuming that the sum of the lengths of the edges in each graph is equal to one, so that the equivalence relation is given by isometry instead of homothety. Each marked graph (g, Γ) belongs to the open simplex of X_n consisting of marked graphs which can be obtained from (g, Γ) by varying the (positive) lengths of the edges, subject to the constraint that the sum of the lengths remain equal to one. If Γ has $k + 1$ edges, the corresponding open simplex of Outer space has dimension k .

Collapsing some edges of Γ to zero corresponds to passing to an (open) face of the simplex containing (g, Γ) . (see Figure 2)

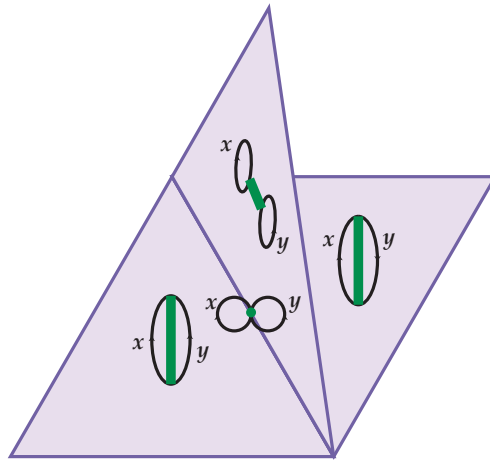


Figure 2: Face identifications

Every open simplex in X_n is a face of a maximal simplex, corresponding to a trivalent graph. An easy Euler characteristic argument shows that a trivalent graph with fundamental group F_n has $3n - 3$ edges; thus the dimension of X_n is equal to $3n - 4$.

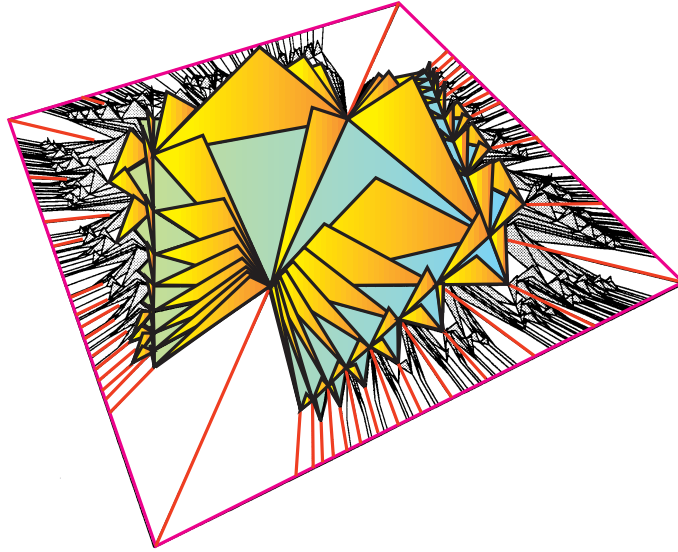


Figure 3: Outer space, $n = 2$

The group $Out(F_n)$ acts on X_n on the right by changing the markings: given $\phi \in Out(F_n)$, choose a representative $f: R_n \rightarrow R_n$ for ϕ ; then $(g, \Gamma)\phi = (g \circ f, \Gamma)$. The stabilizer of a point (g, Γ) is isomorphic to the group of isometries of Γ (see [114]); in particular it is finite.

An edge e of a graph Γ is called a *separating edge* if $\Gamma - e$ is disconnected. There is a natural equivariant deformation retraction of Outer space onto the subspace consisting of marked graphs (g, Γ) such that Γ has no separating edges; the deformation proceeds by uniformly collapsing all separating edges in all marked graphs. This subspace is itself sometimes called Outer space or, if the distinction is important, it is called *reduced Outer space*.

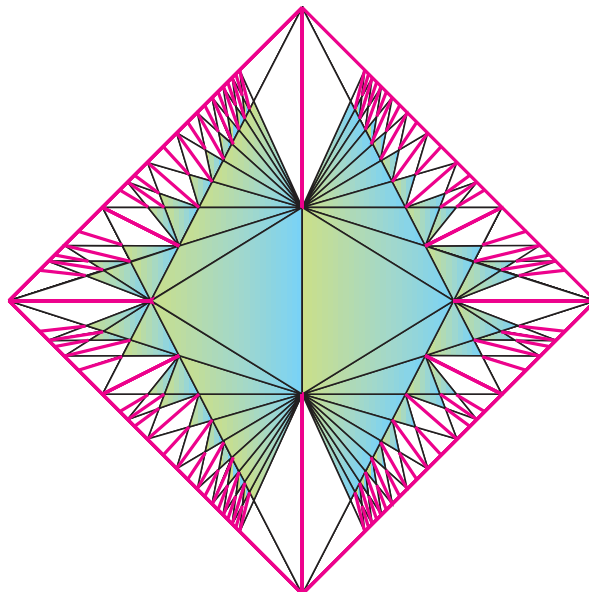


Figure 4: Reduced Outer space, $n = 2$

2. The spine of Outer space

The quotient of X_n by $Out(F_n)$ is not compact. However, X_n contains a *spine* K_n , which is an equivariant deformation retract of X_n whose quotient is compact. This spine K_n has the structure of a simplicial complex, in fact it can be identified with the geometric realization of the partially ordered set of open simplices of X_n . Thus a vertex of K_n is an equivalence class of marked graphs (g, Γ) , considered without lengths on the edges. A set of vertices $\{(g_0, \Gamma_0), \dots, (g_k, \Gamma_k)\}$ spans a k -simplex if (g_i, Γ_i) is obtained from (g_{i-1}, Γ_{i-1}) by collapsing a *forest* in Γ_{i-1} , i.e. a set of edges of Γ_{i-1} which do not contain a cycle.

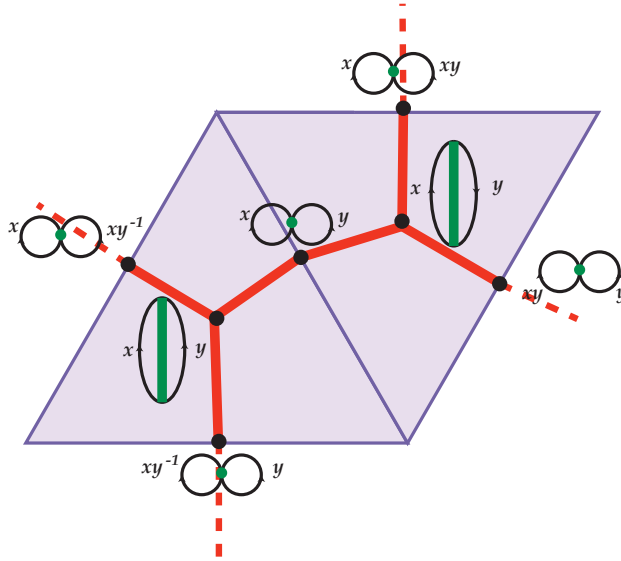


Figure 5: Part of the spine of Outer space, $n = 2$

The same Euler characteristic computation used to find the dimension of Outer space shows that the dimension of K_n is $2n - 3$. Vogtmann [126] showed that K_n is a Cohen-Macaulay complex; in particular, the link of any vertex is homotopy equivalent to a wedge of $(2n - 4)$ -spheres.

Since K_n is contractible and cocompact, and $Out(F_n)$ acts with finite stabilizers, K_n is quasi-isometric to $Out(F_n)$. But in fact the group $Out(F_n)$ can be recovered completely from K_n : the action of $Out(F_n)$ is by simplicial automorphisms, and Bridson and Vogtmann [23] prove that the group of simplicial automorphisms of K_n is precisely equal to $Out(F_n)$.

3. Alternate descriptions and variations

There are several alternate descriptions of Outer space. The most common, and most widely used, is a description as a space of actions on trees. Note that a marking $g: R_n \rightarrow \Gamma$ identifies F_n with $\pi_1(\Gamma)$, thereby giving an action of F_n on the universal cover $\tilde{\Gamma}$ of Γ , which is a tree. The metric on Γ lifts to a metric on $\tilde{\Gamma}$, and F_n acts freely by isometries on $\tilde{\Gamma}$. In the language of \mathbf{R} -trees (see, e.g.[37]),

X_n is the space of free minimal actions of F_n on simplicial \mathbf{R} -trees. For $w \in F_n$, the length of $g(w)$ is the same as the translation length of w along its axis in Γ . The given topology on X_n as a subspace of \mathbf{RP}^C is the same as the “equivariant Gromov-Hausdorff topology” defined independently by Bestvina [7] and Paulin [102] on a set of actions on metric spaces.

Another useful description of Outer space is motivated by work of Whitehead [130]. We fix a doubled handlebody $M = \#(S^1 \times S^2)$ with fundamental group F_n and consider collections $\{s_0, \dots, s_k\}$ of disjointly embedded 2-spheres in M . Such a collection is called a *sphere system* if no sphere s_i bounds a ball in M , and no two spheres s_i and s_j are isotopic for $i \neq j$. We form a simplicial complex $S_{n,0}$ whose k -simplices are isotopy classes of sphere systems in M with exactly $k + 1$ spheres. Taking barycentric coordinates on each simplex corresponds to assigning weights to the spheres, with the sum of the weights in each system equal to one.

A sphere system is *simple* if all of the components of $M - (\bigcup s_i)$ are simply-connected. Outer space can be identified with the subspace of $S_{n,0}$ consisting of isotopy classes of simple sphere systems. To each (weighted) simple sphere system $S = \{s_0, \dots, s_k\}$ we associate a *dual graph* Γ_S , i.e. Γ_S has one vertex for each component of $M - (\bigcup s_i)$ and one edge for each sphere s_i , of length equal to the weight of s_i . To describe the marking on Γ_S , we fix a simple sphere system with exactly n spheres, and let R_n be the graph dual to this system. Both R_n and Γ_S embed into M , and M collapses onto each of them. The marking $g: R_n \rightarrow \Gamma_S$ is given by composing the embedding of R_n with the collapse of M onto Γ_S .

All of these constructions of Outer space can be easily modified to obtain a contractible space on which the group $\text{Aut}(F_n)$ acts with finite stabilizers. (This space was christened “Autre espace” by F. Paulin, but is often anglicized to “Auter space”). In Auter space, a marked graph (g, Γ) comes with a basepoint, which can be either a vertex or an edge point of Γ , and the marking g respects basepoints. To describe Auter space in terms of trees, we replace actions on trees and hyperbolic length functions by actions on rooted trees and Lyndon length functions (see [2]); in terms of sphere systems, we remove a 3-ball from M to obtain $M' = M - B^3$, and consider sphere systems in M' with the additional condition that spheres in a sphere system may not be parallel to the boundary of M' .

4. The boundary and closure of outer space

Motivated by Thurston’s description of the closure of Teichmüller space, we think of Outer space X_n as embedded in the infinite-dimensional projective space \mathbf{RP}^C , and consider its closure \overline{X}_n . A complete and explicit description of \overline{X}_n was given for $n = 2$ in [39], including an embedding of \overline{X}_2 as a 2-dimensional absolute neighborhood retract in Euclidean 3-space. Culler and Morgan proved that \overline{X}_n is compact for all n [37]. Skora [113] and, independently, Steiner [124], showed that \overline{X}_n is contractible.

A finite-dimensional embedding of the closure of Teichmüller space can be obtained by projecting \mathbf{RP}^C onto a finite number of its coordinates; however,

it is shown in [116] that no such projection will result in an embedding of \overline{X}_n , or even of X_n . Nevertheless, Bestvina and Feighn were able to show that \overline{X}_n is finite-dimensional, and in fact has dimension $3n - 4$ [8]. Later Gaboriau and Levitt reproved this result and also showed that the boundary $\partial X_n = \overline{X}_n - X_n$ has dimension $3n - 5$ [46].

All of the results above use the description of Outer space as a space of actions of F_n on simplicial \mathbf{R} -trees. Cohen and Lustig [31] defined an action of F_n on a (not necessarily simplicial) \mathbf{R} -tree to be *very small* if (1) all edge stabilizers are cyclic and (2) for every non-trivial $g \in F_n$ the fixed subtree $Fix(g)$ is isometric to a subset of \mathbf{R} and (3) $Fix(g)$ is equal to $Fix(g^p)$ for all $p \geq 2$. They showed that a simplicial action is in \overline{X}_n if and only if it is very small. Bestvina and Feighn [8] then showed that this is true for all actions, so that the closure of Outer space consists precisely of very small actions of F_n on \mathbf{R} -trees.

The boundary of Outer space contains interesting and unexpected actions. For $n = 2$ all free actions of F_n on \mathbf{R} -trees are simplicial, but for $n \geq 3$ there are free actions on non-simplicial \mathbf{R} -trees in the boundary (see, e.g. [112], [82]). Bestvina and Feighn interpreted much of Rips' theory of actions on \mathbf{R} -trees in terms of *geometric* actions, which are dual to a measured foliations on 2-complexes; but they also found non-geometric actions in the boundary of Outer space [8]. An action in the interior of Outer space is determined by the lengths of finitely many elements of F_n , i.e., given an action, you can find a finite set of conjugacy classes such that no other action produces those lengths for those conjugacy classes ([80] for \mathbf{Z} -actions, [unpub] for simplicial \mathbf{R} -actions); in contrast, there are points on the boundary which are not determined by the lengths of finitely many elements.

Any automorphism α has a fixed point in the closure of X_n (see, e.g., [48], [47], [104]). This means there is an \mathbf{R} -tree T and a homothety $H: T \rightarrow T$, dilating by $\lambda \geq 1$, satisfying $\alpha(w)H = Hw$ for any word $w \in F_n$. Lustig showed that this fixed point may be assumed to have trivial arc stabilizers. Any virtually abelian subgroup of $Out(F_n)$ has a fixed point in the closure of X_n [103].

5. Some other complexes

Contractibility of Outer space is a starting point for showing that several other naturally-defined complexes on which $Out(F_n)$ or $Aut(F_n)$ act are highly connected.

For example, consider the set of all free factorizations $A_1 * \cdots * A_k$ of F_n . Further factorization of the factors A_i defines a partial ordering on this set. The geometric realization of this partially ordered set is homotopy equivalent to a wedge of $(n - 2)$ -dimensional spheres [64]. This is proved by first showing the subspace of Outer space consisting of pointed marked graphs whose basepoint is a cut vertex is $(n - 2)$ -spherical, then showing the map sending each such a graph to the associated splitting of its fundamental group induces a homotopy equivalence of complexes. The fact that this complex is highly connected is useful, for example, in proving homology stability theorems.

A related complex can be formed by considering the set of all free factors of F_n , partially ordered by inclusion. The definition of this complex is analogous to that of the “Steinberg complex” of summands of \mathbf{Z}^n , also partially ordered by inclusion. The classical Solomon-Tits theorem says that the Steinberg complex is $(n - 2)$ -spherical; its top-dimensional homology group can be identified with the Steinberg module for $GL(n, \mathbf{Z})$. Hatcher and Vogtmann [65] show that the free factor complex for F_n is also spherical of dimension $n - 2$, and define the *Steinberg module* for $Aut(F_n)$ to be the top-dimensional homology of this complex.

§2. Algebraic results

1. Finiteness properties

1.1 Finite presentations

$Out(F_n)$ and $Aut(F_n)$ enjoy strong finiteness properties in common with arithmetic groups and mapping class groups. For example, arithmetic and mapping class groups are finitely presented, and in 1924 Nielsen gave finite presentations for both $Aut(F_n)$ and $Out(F_n)$. A simpler finite presentation was given by McCool in 1974 [94].

A particularly simple finite presentation for the index 2 subgroup SA_n of special automorphisms in $Aut(F_n)$ was given by Gersten in 1984 [52] (an automorphism is *special* if its image in $GL(n, \mathbf{Z})$ has determinant 1). Let $X = \{x_1, \dots, x_n\}$ be a free basis for F_n , and $a, b \in X \cup \bar{X}$ with $a \neq b, \bar{b}$. Generators for SA_n are the automorphisms ρ_{ab} sending $a \mapsto ab$, $\bar{a} \mapsto \bar{b}\bar{a}$, and fixing all other elements of $X \cup \bar{X}$. A complete set of relations is:

$$\begin{aligned} \rho_{ab}^{-1} &= \rho_{a\bar{b}} \\ [\rho_{ab}, \rho_{cd}] &= 1 \quad \text{if } a \neq c, d, \bar{d} \text{ and } b \neq c, \bar{c} \\ [\rho_{ab}, \rho_{bc}] &= \rho_{ac} \quad \text{if } a \neq c, \bar{c} \\ w_{ab} &= w_{\bar{a}\bar{b}} \quad (\text{where } w_{ab} = \rho_{ba}\rho_{\bar{a}\bar{b}}\rho_{\bar{b}a}) \\ w_{ab}^4 &= 1 \end{aligned}$$

1.2 Virtual finiteness properties

A finitely presented group always has finitely generated first and second homology groups. Arithmetic and mapping class groups satisfy much stronger homological finiteness properties, which are usually stated as *virtual* finiteness properties, i.e. in terms of torsion-free subgroups of finite index. Specifically, a group G is said to be *WFL* if every torsion-free finite index subgroup H has a free resolution of finite length with each term finitely generated over the group ring $\mathbf{Z}H$. This implies in particular that H has finite cohomological dimension; this dimension is known to be independent of the choice of finite-index subgroup H and is called the *virtual cohomological dimension (VCD)* of G . Arithmetic and mapping class groups are *WFL*.

Baumslag and Taylor [6] showed that the kernel of the natural map from $Aut(F_n)$ to $GL(n, \mathbf{Z})$ is torsion-free, so the inverse image of any torsion-free subgroup of finite index in $GL(n, \mathbf{Z})$ gives a torsion-free finite index subgroup of $Aut(F_n)$ or $Out(F_n)$; thus it makes sense to talk about virtual finiteness properties for these groups. Since the quotient of the spine K_n by $Out(F_n)$ is finite, so is the quotient by any finite index subgroup H ; therefore the chain complex for K_n gives a free resolution of finite length for H with each term finitely generated over $\mathbf{Z}H$, showing that $Out(F_n)$ is *WFL*. Since the dimension of K_n is $2n - 3$, this chain complex has length $2n - 3$, giving an upper bound for the *VCD* of $Out(F_n)$. Similarly, $Aut(F_n)$ is *WFL*, and the *VCD* of $Aut(F_n)$ is at most $2n - 2$. The group $Aut(F_n)$ contains free abelian subgroups of rank $2n - 2$: for example, if $F_n = \langle x_1, \dots, x_n \rangle$, then for $i > 1$ the automorphisms

$$\rho_i: \begin{cases} x_i \mapsto x_i x_1 \\ x_j \mapsto x_j \end{cases} \quad \text{for } j \neq i$$

and

$$\lambda_i: \begin{cases} x_i \mapsto x_1 x_i \\ x_j \mapsto x_j \end{cases} \quad \text{for } j \neq i$$

are a basis for a free abelian subgroup of rank $2n - 2$ in $Aut(F_n)$. The image of this subgroup in $Out(F_n)$ is free abelian of rank $2n - 3$. Since the rank of a free abelian subgroup gives a lower bound for the *VCD*, this shows that the *VCD* of $Out(F_n)$ is exactly $2n - 3$, and the *VCD* of $Aut(F_n)$ is $2n - 2$.

1.3 Residual finiteness

Baumslag (1963, [5]) showed that in general the automorphism group of a residually finite group is residually finite; thus $Aut(F_n)$ is residually finite since F_n is. Grossman (1974, [61]) showed that $Out(F_n)$ too is residually finite. Note also that finitely-generated residually finite groups are automatically hopfian, i.e. any surjective homomorphism from $Aut(F_n)$ or $Out(F_n)$ to itself is an isomorphism.

2. Homology and Euler Characteristic

Hurwicz observed that the homotopy type of an aspherical space depends only on its fundamental group, so that the (co)homology groups of the space are invariants of the group. The cohomology of groups was later defined in purely algebraic terms, and several of the resulting invariants were shown to coincide with invariants which had been previously introduced to study various algebraic problems; for example the first homology $H_1(G)$ is the abelianization of G . In this section we summarize what is known about the cohomology of $Out(F_n)$ and $Aut(F_n)$.

2.1 Low-dimensional calculations

One can see directly from Nielsen's presentation that the abelianization $H_1(Out(F_n))$ is \mathbf{Z}_2 for $n > 2$. The homology of $Out(F_2) = GL(2, \mathbf{Z})$ is well known; it can be computed from the amalgamated free product decomposition

$$GL(2, \mathbf{Z}) = D_8 *_{(\mathbf{Z}_2 \times \mathbf{Z}_2)} (S_3 \times \mathbf{Z}_2)$$

using the Mayer-Vietoris sequence (see Brown's book [26], Exercise 3, p. 52 for $SL(2, \mathbf{Z})$). Gersten [52] computed that $H_2(\text{Aut}(F_n)) = \mathbf{Z}_2$ for $n \geq 5$ and clarified the relation with $H_2(SL(n, \mathbf{Z}))$ and $K_2(\mathbf{Z})$ by considering a “non-abelian” version of the Steinberg group.

Since Outer space and its spine K_n are contractible, and since $\text{Out}(F_n)$ acts with finite stabilizers, the quotient of K_n by any torsion-free subgroup Γ of finite index is an aspherical space, and the homology of this space is equal to the homology of Γ . In fact, even though the action of $\text{Out}(F_n)$ on K_n is not free, the homology of $\text{Out}(F_n)$ can be computed using the equivariant cohomology spectral sequence associated to the action; to do this, one must have a detailed description of the cell structure of the quotient of K_n by $\text{Out}(F_n)$ and know the cohomology of the stabilizers of all cells. This approach was used by Brady [18], who completely calculated the integral cohomology of $\text{Out}_+(F_3)$, where the “+” denotes the preimage of $SL(3, \mathbf{Z})$.

2.2 Homology Stability

Computations for small values of n can be leveraged to those for high values via “homology stability” theorems, which say that for n sufficiently large with respect to i , the i th homology group H_i is independent of n . Hatcher [62] showed that the inclusion $\text{Aut}(F_n) \rightarrow \text{Aut}(F_{n+1})$ induces an isomorphism on the i -th homology group for $n > (i^2/4) + 2i - 1$, and also that the natural projection induces an isomorphism $H_i(\text{Aut}(F_n)) \cong H_i(\text{Out}(F_n))$ for $n > (i^2/4) + (5i/2)$. His methods are similar to methods used by Harer to prove homology stability for mapping class groups of surfaces, which were in turn inspired by methods of Quillen and others to prove homology stability for various classes of linear and arithmetic groups. Harer proves homology stability by considering equivariant spectral sequences arising from the action of the mapping class group on various complexes of curves and arcs on a surface. In Hatcher's work, these are replaced by complexes of 2-spheres imbedded in a connected sum of $S^1 \times S^2$'s; in particular, one of these is the complex $S_{n,0}$ which contains Outer space.

2.3 Cerf theory, rational homology and improved homology stability

In [64] Hatcher and Vogtmann define the *degree* of a basepointed graph with fundamental group F_n to be $2n - |v|$, where $|v|$ is the valence of the basepoint. They develop an analog of Cerf theory for parameterized families of basepointed graphs, in order to prove that the subspace D_k of Outer space consisting of marked basepointed graphs of degree at most k is $(k - 1)$ -connected. Thus D_k can be thought of as a kind of k -skeleton for Outer space. The action of $\text{Aut}(F_n)$ on Outer space restricts to an action on D_k , and they use the spectral sequence arising from this action to improve the homology stability results mentioned above. Specifically, they show that the natural inclusion $\text{Aut}(F_n) \rightarrow \text{Aut}(F_{n+1})$ induces an isomorphism $H_i(\text{Aut}(F_n)) \cong H_i(\text{Aut}(F_{n+1}))$ for $n \geq 2i + 3$. An even stronger statement is obtained for homology with trivial rational coefficients: $H_i(\text{Aut}(F_n); \mathbf{Q}) \cong H_i(\text{Aut}(F_{n+1}); \mathbf{Q})$ for $n \geq 3i/2$. This stronger statement follows because the quotient of D_k by $\text{Aut}(F_n)$ is independent of n for n large with

respect to k ; in particular, the argument is purely geometric and does not depend on analyzing a spectral sequence. In [66] a refinement of this geometric argument is exploited to give a stability range of $n \geq 5i/4$, and the rational homology in dimensions $i \leq 7$ is computed for all values of n . The only non-zero homology group in this range is $H_4(\text{Aut}(F_4); \mathbf{Q}) = \mathbf{Q}$ [66].

The Cerf theory techniques used to study Outer space depend on the fact that the marked graphs in question have basepoints. In particular, they do not apply to Outer space, and the stability range for the homology of $\text{Out}(F_n)$ has not yet been improved from the original quadratic bound, though analogies with mapping class groups and linear groups as well as the result for $\text{Aut}(F_n)$ suggest that the bound should be linear rather than quadratic. Meanwhile, the computational techniques used to calculate the rational homology of $\text{Aut}(F_n)$ for low values of n can be adapted for $\text{Out}(F_n)$, and Vogtmann [unpub] has shown that the class $H_4(\text{Aut}(F_4); \mathbf{Q})$ survives in $\text{Out}(F_4)$, and in fact $H_4(\text{Out}(F_4); \mathbf{Q}) = \mathbf{Q}$.

2.4 Kontsevich's Theorem

Kontsevich [73] has related the cohomology of $\text{Out}(F_n)$ with coefficients in a field of characteristic zero to the homology of a certain Lie algebra $\ell_\infty = \lim_{n \rightarrow \infty} \ell_n$. The algebra ℓ_n consists of the derivations of the free Lie algebra on $2n$ generators which kill a basic ‘‘symplectic element.’’ Using this correspondence, Morita [98] has produced a cycle representing the non-trivial homology class in $H_4(\text{Out}(F_4); \mathbf{Q})$. His construction in fact gives cycles in dimension $4k$ for $\text{Out}(F_{2k+2})$ for all $k \geq 1$, but it is unknown whether these cycles are trivial in homology for $k > 1$. Morita also produces a map from the positive part ℓ_∞^+ of ℓ_∞ to an abelian Lie algebra, which he conjectures is the abelianization of ℓ_∞^+ ; if this is correct, this implies that the top-dimensional rational homology $H_{2n-3}(\text{Out}(F_n); \mathbf{Q})$ is trivial for all n . Using computational techniques borrowed from the work of Hatcher and Vogtmann for $\text{Aut}(F_n)$, Vogtmann [unpub] has verified that this homology does vanish for $n \leq 7$.

2.5 Torsion

Since the homology of $\text{Aut}(F_n)$ is finitely generated, homology stability for $\text{Aut}(F_n)$ shows that the direct limit Aut_∞ of the groups $\text{Aut}(F_n)$ has finitely generated integral homology in all dimensions. Homology classes in $H_i(\text{Aut}(F_n))$ which do not survive in $H_i(\text{Aut}_\infty) = \lim_{k \rightarrow \infty} H_i(\text{Aut}(F_k))$ are called *unstable classes*, while the homology of $H_i(\text{Aut}_\infty)$ is called the *stable homology*. Though calculations to date have produced no stable rational classes, it is known that the stable integral homology does contain many torsion classes. This is shown by considering certain spaces related to algebraic K -theory. Specifically, we note that the commutator subgroup E_∞ of Aut_∞ is a perfect normal subgroup, so that one can form the associated plus construction $B\text{Aut}_\infty \rightarrow B\text{Aut}_\infty^+$. This space $B\text{Aut}_\infty^+$ is a space with fundamental group $\text{Aut}_\infty/E_\infty$ and the same homology as Aut_∞ . Cohen and Peterson [29] first noticed that $B\text{Aut}_\infty^+$ can be decomposed as a product, where one of the factors can be identified, after inverting the prime 2

at least, with the classical homotopy-theoretic space ImJ , whose mod p homology contains the mod p homology of $GL(\mathbf{F}_q)$ for many primes q (here \mathbf{F}_q is the field with q elements). ImJ is a subspace of the space $B\Sigma_\infty^+$, and using results of Waldhausen, Hatcher [62] later showed that in fact $BAut_\infty^+$ contains the entire space $B\Sigma_\infty^+$ as a direct factor, so that the integral homology of the infinite symmetric group Σ_∞ is a direct summand of $H_*(Aut_\infty)$.

The effect of torsion in a group G on the cohomology of G is measured by Farrell cohomology, and above the virtual cohomological dimension of G the Farrell cohomology coincides with the ordinary cohomology (see Brown's book [26]). For an odd prime p , Glover, Mislin and Voon [56] compute the p -part of the Farrell cohomology of $Aut(F_{p-1})$ and $Out(F_{p-1})$ to be $\mathbf{F}_p[w, w^{-1}]$, with w of degree $2(p-1)$ [56]. In $Aut(F_{p-1})$ and $Out(F_{p-1})$ all p -subgroups are cyclic of order p , and all are conjugate to a fixed p -subgroup P . This implies that the Farrell cohomology is the same as the Farrell cohomology of the normalizer $N(P)$, which is finite and maps onto the "holomorph" $P \rtimes Aut(P)$. The kernel of this map has order prime to p , so that the p -part of the Farrell cohomology of $N(P)$ is equal to that of $P \rtimes Aut(P)$, which is easy to compute. In [55], the action of $Out(F_n)$ on the spine of Outer space was used to compute the Farrell cohomology of $Out(F_n)$ for $n = p-1, p$ and $p+1$; these methods were extended by Chen [28] to compute the Farrell cohomology in other cases where all of the p -subgroups are cyclic of order p , including $Out(F_{p+2})$, $Aut(F_{p+1})$ and $Aut(F_{p+2})$. Using different methods, Jensen [68] considers the first case where p -subgroups are not cyclic, and manages to compute the Farrell cohomology of $Aut(F_n)$ for $n = 2(p-1)$.

2.6 Euler characteristic

The rational Euler characteristic $\tilde{\chi}(G)$ of a *WFL* group G is defined as the (standard) Euler characteristic of any torsion-free finite index subgroup Γ , divided by the index of Γ . This is a well-defined group invariant by a theorem of Swan, which says that $\tilde{\chi}(G)$ is independent of the choice of Γ (see Brown [26]). For arithmetic groups and mapping class groups, there are close connections between the denominators of the rational Euler characteristics and classical number theory. For $Out(F_n)$, Smillie and Vogtmann found a generating function for the rational Euler characteristic by analyzing the cells and stabilizers of the action of $Out(F_n)$ on the spine K_n of Outer space [114]. Using this generating function, they computed values of $\tilde{\chi}(Out(F_n))$ for all values of $n \leq 100$. The results suggest that the Euler characteristic is always negative and that its absolute value grows more than exponentially with n . In a subsequent paper, they computed the p -part of the denominator for infinitely many values of p and n [115]. For $p = 2$, these computations imply in particular that $\tilde{\chi}(Out(F_n))$ is non-zero for all even values of n .

There is an interesting application of these calculations to the kernel IA_n of the natural map from $Out(F_n)$ to $GL(n, \mathbf{Z})$. In 1934, Magnus [90] showed that IA_n is finitely generated and asked whether it was finitely presentable for any $n \geq 3$. Note that a finitely presented group has finitely generated second

homology. If the homology of IA_n were finitely generated in all dimensions, the short exact sequence $1 \rightarrow IA_n \rightarrow \text{Out}(F_n) \rightarrow GL(n, \mathbf{Z}) \rightarrow 1$ would imply that $\tilde{\chi}(\text{Out}(F_n)) = \tilde{\chi}(IA_n)\tilde{\chi}(GL(n, \mathbf{Z}))$. However, the rational Euler characteristic of $GL(n, \mathbf{Z})$ is known to be zero, so in cases where we know $\tilde{\chi}(\text{Out}(F_n))$ to be non-zero (e.g. for $n < 100$ or for any n even), we have a contradiction. In particular, for $n = 3$ the cohomological dimension of IA_3 is equal to 3, and $H_1(IA_3)$ is finite, so this implies that either $H_2(IA_3)$ or $H_3(IA_3)$ is not finitely generated. In fact, Krstić and McCool [79] have answered Magnus' question for $n = 3$, showing that IA_3 is not finitely presentable (and hence $H_2(IA_3)$ is not finitely generated). It is still unknown whether $H_3(IA_3)$ is finitely generated, or whether IA_n is finitely presentable for $n \geq 4$.

3. Ends and virtual duality

End invariants of a topological space can be thought of as measuring the topology of the space outside arbitrarily large compact sets. If a group G acts properly and cocompactly on a contractible space X , the end invariants of X are invariants of G , often called invariants “at infinity.” Examples include the number of ends, the fundamental group at infinity, and the degree of connectivity at infinity. The possible values for end invariants of a group are generally more limited than for those of an arbitrary topological space; for example a topological space can have any number of ends, but a group can have only 0, 1, 2 or infinitely many ends, by a celebrated theorem of Hopf.

One can attempt to compute end invariants of $\text{Out}(F_n)$ by computing end invariants of the spine K_n of Outer space. This approach was taken in [125] to show that $\text{Out}(F_n)$ has one end for $n \geq 3$, and is simply connected at infinity for $n \geq 5$. J. Rickert (1995, [107]) later extended this to show that $\text{Out}(F_4)$ is simply-connected at infinity.

Bestvina and Feighn [9] used a different approach, using reduced Outer space rather than its spine, to show that $\text{Out}(F_n)$ is in fact $(2n - 5)$ -connected at infinity for all n . They did this by constructing a “Borel Serre bordification” of reduced Outer space; this is a space Y containing reduced Outer space as its interior, such that the action of $\text{Out}(F_n)$ extends to Y , and the quotient of Y by the action is compact. They prove that Y is $(2n - 5)$ -connected at infinity (and hence that $\text{Out}(F_n)$ is $(2n - 5)$ -connected at infinity) by constructing and analyzing a Morse function on Y which measures the lengths of conjugacy classes of elements of F_n at different points of Y .

This result was motivated by the following algebraic consequence. A group is said to be a *virtual duality group* of dimension d if it has a finite-index subgroup H which is a d -dimension duality group, i.e. there is an H -module D and for each i an isomorphism

$$H^i(H; M) \cong H_{d-i}(H, D \otimes M)$$

for any H -module M . Borel and Serre proved that arithmetic groups are virtual duality groups, and Harer proved that mapping class groups are virtual duality

groups. By results of Bieri and Eckmann, G is a d -dimensional virtual duality group if and only if $H^i(G, \mathbf{Z}G)$ vanishes in all dimensions except d , where it is torsion-free. $H^i(\text{Out}(F_n), \mathbf{Z}[\text{Out}(F_n)])$ is closely related to the cohomology with compact supports of Y , and hence to the degree of connectivity at infinity of Y . Specifically, the fact that $\text{Out}(F_n)$ is $2n - 5$ -connected at infinity implies, by general theory (see [50]), that $\text{Out}(F_n)$ is a virtual duality group of dimension $2n - 5$.

4. Fixed subgroup of an automorphism

Given an automorphism α of F_n , the set of elements of F_n fixed by α form a group called the *fixed subgroup* of α . Dyer and Scott [43] showed that the fixed subgroup of a finite-order automorphism is either cyclic or is a free factor of F_n . In particular, the fixed subgroup has rank at most n , and it was conjectured that this rank statement is true for any automorphism (the Scott Conjecture). Gersten showed that the fixed subgroup of any automorphism is finitely generated [53]; elegant proofs of this were given by Goldstein and Turner [57, 58] and by Cooper [35]. In their 1992 Annals paper, Bestvina and Handel [10] proved the Scott Conjecture using their theory of train tracks; other proofs and generalizations using the theory of group actions on \mathbf{R} -trees were given by Sela in [109], by Gaboriau, Levitt and Lustig in [48], and by Paulin in [104].

Bestvina and Handel's train track theory is modelled on Thurston's theory of train tracks on surfaces. A *topological representative* of an outer automorphism ϕ of F_n is a marked graph (g, Γ) together with a homotopy equivalence $f: \Gamma \rightarrow \Gamma$ such that the map induced by $g^{-1}fg$ on fundamental groups is equal to ϕ ; here the homotopy equivalence f must take vertices to vertices and be an immersion on each edge. The marking g is usually suppressed in the notation, as changing g simply conjugates ϕ by an element of $\text{Out}(F_n)$; thus g plays little role in the analysis of the automorphism. A *train track map* is a topological representative which is completely taut, in the sense that the image of a single edge of Γ is an immersion under all iterates of f .

It is not always possible to find a train track map for an outer automorphism ϕ , but Bestvina and Handel prove that it is possible if ϕ is *irreducible*. An automorphism is *reducible* if it has a topological representative $f: \Gamma \rightarrow \Gamma$ such that Γ contains a proper invariant subgraph Γ_0 which is not a forest; it is irreducible if it is not reducible. The notion of irreducibility can also be defined using the *transition matrix* of a topological representative $f: \Gamma \rightarrow \Gamma$; this matrix has (i, j) -entry equal to k if the image of the j th edge of Γ crosses the i th edge exactly k times (in either direction). The topological representative f is irreducible if and only if its transition matrix is irreducible. An automorphism ϕ is irreducible if and only if every topological representative of ϕ is irreducible.

To show that an automorphism is reducible, it suffices to find one topological representative with the right properties. It is trickier to show that an automorphism is irreducible. Stallings and Gersten, in [123] found large classes of explicit

examples of irreducible automorphisms, including automorphisms which are irreducible and all of whose powers are irreducible.

If $f: \Gamma \rightarrow \Gamma$ is an irreducible train track map, the transition matrix for f has a unique largest positive eigenvalue λ . If the edges of Γ are given lengths equal to the entries of the eigenvector for λ , then f stretches each edge by a factor of λ . Thus each irreducible train track map corresponds naturally to a point in Outer space, namely the marked graph underlying the topological representative f with edge lengths given by the entries of the eigenvector for λ .

Any automorphism of a punctured surface induces an automorphism of the free fundamental group F_n of the surface; such an automorphism of F_n is called *geometric*. Stallings showed that not every automorphism of F_n is geometric [119]: an automorphism of a surface preserves the intersection form on the surface, so the map induced on homology has eigenvalues occurring in conjugate pairs. Stallings constructed automorphisms of free groups with eigenvalues which do not have this property. If an automorphism ϕ is irreducible Bestvina and Handel showed that the rank of the fixed subgroup of ϕ is at most 1; if it is equal to 1 and all powers of ϕ are irreducible, then ϕ is geometric, realizable on a surface with one puncture [10].

Although a general automorphism may not have a train track representative, there is always a filtered topological representative f , called a *relative train track*, which has the following properties: f restricted to the lowest level of the filtration is a train track, and f maps each level of the filtration to the same or lower levels, in a very controlled manner. Bestvina and Handel analyze these relative train tracks to prove the Scott Conjecture.

In [34] Collins and Turner apply train track methods to give an explicit description of all automorphisms whose fixed subgroup has maximal rank, i.e. rank exactly n . In particular, any such automorphism has linear growth.

In [47], Gaboriau, Jaeger, Levitt and Lustig give a new proof of the Scott Conjecture from the point of view of group actions on trees. They obtain more information on the rank of the fixed subgroup: for an automorphism $\alpha: F_n \rightarrow F_n$ they get the formula

$$\text{rank}(\text{Fix}(\alpha)) + \frac{1}{2}a(\alpha) \leq n,$$

where $a(\alpha)$ is the number of orbits under $\text{Fix}(\alpha)$ of attracting fixed points on the boundary of F_n , i.e. the space of ends of the Cayley graph of F_n .

5. The conjugacy problem

Since the group $\text{Out}(F_2)$ is isomorphic to $GL(2, Z)$, it has a solvable conjugacy problem.

Solvability of the conjugacy problem for finite-order automorphisms, and in fact for finite subgroups of $\text{Out}(F_n)$, follows from work of Krstić [76] or Kaladjevsky [69]. For finite subgroups, one can also decide whether there is a conjugating automorphism which takes a given set S of cyclic words to another given set

S' ; this generalizes a result of Whitehead which gives an algorithm for deciding whether there is *any* automorphism taking S to S' [78].

A *Dehn twist automorphism* of a free group is a specific type of automorphism which can be described in terms of a graph-of-groups decomposition of the free group with cyclic edge groups. The class of Dehn twist automorphisms includes automorphisms induced from Dehn twists on (punctured) surfaces. An algorithm for deciding whether two Dehn twist automorphisms are conjugate was given by Cohen and Lustig in [32]. Results from this paper were combined with the Whitehead algorithm mentioned above to decide whether two “roots” of Dehn twists are conjugate in [78].

In [108] Sela applies techniques from his solution of the isomorphism problem for hyperbolic groups to solve the conjugacy problem for irreducible automorphisms. Los [84] has also published a solution to the conjugacy problem for irreducible automorphisms, based on the train track techniques of Bestvina and Handel.

M. Lustig and Z. Sela [109], [110] have described normal forms, or prime decompositions, for automorphisms of free groups. Lustig has given a complete solution of the conjugacy problem for $Out(F_n)$ [86, 87].

6. Subgroups

6.1 Finite subgroups and their centralizers

The Realization Theorem says that any finite subgroup of $Aut(F_n)$ or $Out(F_n)$ can be realized as a group of automorphisms of a graph with fundamental group F_n (see Culler [36] or Zimmerman [132] or Khramtsov [71]). The proof goes as follows: given a subgroup Γ of $Out(F_n)$, the inverse image in $Aut(F_n)$ is an extension of F_n by Γ , i.e. it is a free-by-finite group, so acts on a tree T with finite stabilizers by the theorem of Karass, Pietrowski and Solitar [70]. The quotient T/F_n is a graph with fundamental group F_n which inherits an action by Γ , and it is this graph and action which realize Γ . If Γ is the (isomorphic) image of a subgroup of $Aut(F_n)$, that subgroup fixes a point of T (since any finite group acting on a tree has a fixed point), so that Γ is realized by a group of automorphisms of a pointed graph.

The Realization Theorem can be used to classify finite subgroups of $Aut(F_n)$ or $Out(F_n)$; for example, maximal order p -subgroups were classified in [114]. The maximal order of a finite subgroup of $Out(F_n)$ is $2^n n!$ [127], and is realized as the group of automorphisms of the standard rose R_n . For $n = 3$, the stabilizer of the complete graph on 4 vertices also has order $2^3 3! = 24$, but for $n > 3$, any subgroup of order $2^n n!$ is the stabilizer of some rose (see Kulkarni [81] or Wang-Zimmermann [127]).

One can also determine the maximum orders of finite cyclic subgroups of $Aut(F_n)$ using the Realization Theorem. Finite-order automorphisms in F_n with $n \leq 5$ were classified by Krstić [75]. Kulkarni characterized finite-order automorphisms of F_n for all n as follows [81]: Let ϕ be the Euler phi-function, m_p be the p -primary part of m , and define $k(m) = \sum_p \phi(m_p)$. If $m = 2p$ for some odd

prime p , then the orders of finite cyclic subgroups of $Out(F_n)$ are the integers m such that $k(m) \leq n + 1$; otherwise, the orders are the m such that $k(m) \leq n$.

Work of Levitt and Nicolas [83], and independently of Bao [4], shows that the maximal order of elements in $Aut(F_n)$ grows asymptotically like $\exp(\sqrt{n \log n})$, the same asymptotic formula as for the symmetric group and for the general linear group $GL(n, \mathbf{Z})$. They also show that the numbers 2, 6 and 12 are the only values of n for which the maximal order of a finite-order element of $Aut(F_n)$ is different from that in $GL(n, \mathbf{Z})$. These are also the only even values for which the maximal order in $Aut(F_n)$ is different from that in $Aut(F_{n+1})$; this is related to the fact that for these values of n the maximal order $G(n)$ in $GL(n, \mathbf{Z})$ cannot be realized by a block matrix associated to the factorization of $G(n)$ into prime powers.

In terms of the spine K_n of Outer space, the Realization Theorem says that a finite subgroup of $Out(F_n)$ fixes a vertex of K_n . Since the quotient of K_n by $Out(F_n)$ is finite, this shows that there are only finitely many conjugacy classes of finite subgroups.

Torsion subgroups of $Out(F_n)$ are finite: take any normal torsion free subgroup Γ of finite index; the map from $Out(F_n)$ to $H = Out(F_n)/\Gamma$ sends any torsion subgroup injectively into the finite group H .

If G is a finite subgroup of $Out(F_n)$, let $C(G)$ denote the centralizer of G in $Out(F_n)$. Krstić showed that $C(G)$ is finitely presented [77]. Let K_G denote the subcomplex of K_n fixed by G ; then $C(G)$ acts on K_G . K_G is contractible, and has an equivariant deformation retract L_G which is, in general, of lower dimension. The dimension of L_G can be computed for many G [80], and gives bounds on the VCD of $C(G)$. The quotient of L_G by $C(G)$ is finite, showing that $C(G)$ is WFL. Boutin [17] and Pettet [105] have given elementary criteria for determining exactly when the VCD is zero, i.e. when $C(G)$ is finite.

Results of McCool relate centralizers of finite subgroups to automorphism groups of free-by-finite groups [95]; this connection is used in [80] to obtain bounds on the VCD of the group of automorphisms of a free-by-finite group. In particular, the outer automorphism group of a free product of n finite groups has VCD equal to $n - 2$. McCullough and Miller [97] generalized the construction in [38] to find a contractible complex on which the outer automorphism group of a general free product of groups acts. Their methods give a different proof that the VCD of a free product of n finite groups is equal to $n - 2$.

6.2 Stabilizers, mapping class groups and braid groups

To prove contractibility of Outer space, Culler and Vogtmann define an integer-valued norm on vertices of the spine K_n and use this norm as a kind of Morse function to show that balls of any radius in K_n are contractible. This norm is defined by measuring the lengths of a certain fixed set of cyclic words in F_n . However, any finite set W of cyclic words can be used in a similar way to define a norm, and it follows from the proof that balls in any such norm are contractible. Fix such a finite set W , and let K_W denote the ball of minimal

positive radius with respect to W . Then K_W is invariant under the action of the stabilizer of W in $Out(F_n)$, so acts as a natural geometry for the stabilizer; for example, the dimension of K_W bounds the VCD of the stabilizer. One example of particular interest is the case where W is the set of peripheral elements for a surface S with punctures; the stabilizer of W is then the mapping class group $M(S)$ of that surface, and the dimension of K_W is equal either to the VCD of $M(S)$ or the VCD+1. Other interesting examples are $W = \{x_1, \dots, x_n, x_1x_2 \dots x_n\}$ (which contains the pure braid group on n strands) and $W = \{x_1, \dots, x_n\}$ (the group of “symmetric” automorphisms).

6.3 Abelian subgroups, solvable subgroups and the Tits alternative

J. Tits established, for any linear group G , the remarkable dichotomy that subgroups of G are either virtually solvable or they contain a non-abelian free group; this property has become known as the *Tits alternative*. N. Ivanov [67] and J. McCarthy [93] proved that the Tits alternative holds for mapping class groups of surfaces. Ivanov [67] and Birman, Lubotzky and McCarthy [15] showed further that (virtually) solvable subgroups of mapping class groups are virtually abelian, and that abelian subgroups are finitely generated, of rank bounded by a linear function of the genus of the surface.

Bass and Lubotzky [3] showed that all abelian subgroups of $Out(F_n)$ are finitely generated by considering a “linear-central filtration” for $Out(F_n)$. Since the VCD of $Out(F_n)$ is equal to $2n - 3$, this is also a bound on the Hirsch rank of $Out(F_n)$, and hence on the rank of a finitely-generated abelian subgroup.

The Tits alternative for $Out(F_n)$ was proved by Bestvina, Feighn and Handel in a series of two papers [12, 13]. The first paper deals with subgroups which contain an exponentially growing automorphism, and contains a detailed analysis of the dynamics of the action of $Out(F_n)$ on a certain space of *laminations* of F_n . A special case of this (when the subgroup contains an irreducible automorphism) is dealt with in [11], where it is also shown that if a solvable subgroup contains an irreducible automorphism then it is virtually cyclic. The second paper deals with subgroups all of whose elements grow polynomially, and contains a graph-theoretic analog of Kolchin’s theorem for linear groups, which says that a subgroup consisting entirely of unipotent elements is conjugate to a subgroup of upper-triangular matrices. The second paper studies the action of $Out(F_n)$ on the closure of Outer space to find a virtual fixed point for the subgroup. In a fourth paper [14], the same authors show that solvable subgroups of $Out(F_n)$ are virtually abelian.

Feighn and Handel have recently developed this theory further to describe all abelian subgroups of $Out(F_n)$ in terms of train track representatives for certain elements of the subgroups.

7. Rigidity properties

Lattices in semisimple Lie groups enjoy “rigidity” properties saying that maps between lattices extend to special kinds of maps between the ambient Lie

groups. For example, any isomorphism between arithmetic lattices in semisimple \mathbf{Q} -algebraic groups extends to a rational isomorphism of the groups. This severely limits the possibilities for automorphisms of a lattice, generalizing the classical fact that the only automorphisms of $GL(n, \mathbf{Z})$ are inner automorphisms, the map sending a matrix to its transpose inverse and, for n even, the map multiplying a matrix by its determinant.

In [42], Dyer and Formanek used a classical result of Burnside on characteristic subgroups to show that all automorphisms of $Aut(F_n)$ are inner. Khramtsov gave another proof of this fact, as well as proving the same result for $Out(F_n)$ [72]. Bridson and Vogtmann [24] give new proofs for both $Aut(F_n)$ and $Out(F_n)$ by considering the actions of these groups on Outer space and Outer space.

In the classical setting, Margulis super-rigidity tells one that if there is no homomorphism from one semi-simple group G_1 to another G_2 , then any map from a lattice in G_1 to G_2 must have finite image. For example, if $m < n$ then any map from $GL(n, \mathbf{Z})$ to $GL(m, \mathbf{Z})$ has image Z_2 or $\{1\}$. The analogous statements for $Out(F_n)$ and $Aut(F_n)$ are proved in [25], as a corollary of the stronger statement that any map from $Out(F_n)$ to a group which does not contain the symmetric group Σ_{n+1} must have image of order at most 2. Another result in this direction is the fact that if Γ is an irreducible, non-uniform lattice in a higher rank semisimple group, then any homomorphism from Γ to $Aut(F_n)$ or $Out(F_n)$ has finite image; this was noted in [21], and follows by combining the main theorem of [13] together with Margulis' characterization of normal subgroups of such Γ .

8. Relation to other classes of groups

8.1 Arithmetic and linear groups

Many properties we have established for $Out(F_n)$ and $Aut(F_n)$ are general properties of linear groups, and one might be led to wonder whether $Aut(F_n)$ or $Out(F_n)$ is in fact linear, or perhaps even arithmetic.

$Aut(F_n)$ and $Out(F_n)$ are not arithmetic for $n \geq 3$. The following argument of N. Ivanov [talk at MSRI, 1995] for mapping class groups of surfaces also works for automorphism groups of free groups. Both $Out(F_n)$ and $Aut(F_n)$ contain subgroups of the form $F_2 \times \mathbf{Z}$, showing that they cannot be rank one arithmetic groups. The short exact sequences

$$1 \rightarrow IA_n \rightarrow Out(F_n) \rightarrow GL(n, \mathbf{Z}) \rightarrow 1$$

and

$$1 \rightarrow JA_n \rightarrow Aut(F_n) \rightarrow GL(n, \mathbf{Z}) \rightarrow 1$$

show that they are also not rank ≥ 2 , since Margulis showed that any normal subgroup of a higher rank arithmetic group is either central or has finite index; but $GL(n, \mathbf{Z})$ is not finite, and the kernels IA_n and JA_n are not central.

Formanek and Procesi [45] showed that $Aut(F_n)$ ($n \geq 3$) and $Out(F_n)$ ($n \geq 4$) have no faithful linear representations. They did this by exhibiting a subgroup

of $Aut(F_3)$ with properties which cannot hold in a linear group. This subgroup H is generated by the following five automorphisms of $F_3 = F\langle x_1, x_2, x_3 \rangle$:

$$\phi_1: \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_2 \\ x_3 \mapsto x_3x_1 \end{cases} \quad \phi_2: \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_2 \\ x_3 \mapsto x_3x_2 \end{cases} \quad \alpha_i: z \mapsto x_i z x_i^{-1}$$

The automorphisms ϕ_1 and ϕ_2 generate a free group of rank two, as do the inner automorphisms α_1 and α_2 , and these two free groups commute. The subgroup H is an HNN extension of $F\langle \phi_1, \phi_2 \rangle \times F\langle \alpha_1, \alpha_2 \rangle$ by $t = \alpha_3$ of the form

$$\langle G \times G, t|t(g, g)t^{-1} = (1, g) \text{ for all } g \in G \rangle .$$

Formanek and Procesi show that if such an HNN extension is linear, then G must be virtually nilpotent; but in our case G is free, so is not virtually nilpotent. Since $Aut(F_3)$ imbeds in $Aut(F_n)$ ($n \geq 3$) and $Aut(F_n)$ imbeds in $Out(F_{n+1})$, this proves the theorem.

The group $Out(F_2) \cong GL(2, \mathbf{Z})$ is of course linear. Linearity of $Aut(F_2)$ is equivalent to the linearity of the 4-strand braid group B_4 , which was proved by Krammer [74]. It is still unknown whether $Out(F_3)$ is linear.

8.2 Automatic and hyperbolic groups

The group $Out(F_2) = GL(2, \mathbf{Z})$ is hyperbolic, since it has a free subgroup of finite index. (Consider the standard action of $GL(2, \mathbf{Z})$ on a trivalent tree, with finite stabilizers. Any torsion-free subgroup of finite index acts freely on the tree, so is free). The groups $Aut(F_n)$, $n \geq 2$ and $Out(F_n)$, $n \geq 3$ are not hyperbolic since they contain free abelian subgroups of rank 2.

The group $Aut(F_2)$ is automatic. This follows from the fact that the braid group B_n is automatic; this implies that B_n/Z_n is automatic, where Z_n is the (cyclic) center of B_n [44] and from the fact that B_4/Z_4 is isomorphic to a finite-index subgroup of $Aut(F_2)$ [41]. Brady [19] has written down an explicit automatic structure for a finite-index subgroup of $Aut(F_2)$. More precisely, he describes a 2-dimensional CAT(0) simplicial complex on which a finite-index subgroup of $Aut(F_2)$ acts freely and cocompactly. This does not apparently give a biautomatic structure, however.

Hatcher and Vogtmann showed that $Aut(F_n)$ and $Out(F_n)$ satisfy exponential isoperimetric inequalities [63]. For $n = 3$ this is best possible; this is shown by exhibiting a loop in $Aut(F_3)$ which maps to a loop in $GL(3, \mathbf{Z})$ which is known to be exponentially hard to fill. In particular, this implies that $Aut(F_3)$ and $Out(F_3)$ are not automatic. By considering certain centralizers in $Aut(F_n)$ and $Out(F_n)$, Bridson and Vogtmann showed that for $n \geq 3$, neither $Aut(F_n)$ nor $Out(F_n)$ can act on a CAT(0) space properly and cocompactly. (Bridson [20] had originally shown that the spine K_n of Outer space cannot be given a piecewise-Euclidean metric of non-positive curvature, for $n \geq 3$. Gersten proved that $Out(F_n)$ cannot act on a CAT(0) space for $n \geq 4$, using work of Bridson on flat subspaces of CAT(0) spaces). Bridson and Vogtmann also show that $Aut(F_n)$ and $Out(F_n)$ are

not semihyperbolic, in the sense of Alonso and Bridson, and in particular they are not biautomatic [22]. It is unknown whether they are automatic or asynchronously automatic.

9. Actions on trees and Property (T)

A group is said to have Serre's property FA if every action on a tree has a global fixed point. A stronger property, which implies FA, is Kazdan's property (T), which says that the trivial representation is isolated in the space of unitary representations with an appropriate topology. The fact that (T) implies FA was first proved by Watatani [128], and generalized to any locally compact group (discrete, p-adic, Lie etc) by Alperin [1]. Serre proved that property (T) is equivalent to the fixed point property for actions on Hilbert spaces [unpub].

$Out(F_n)$ has property FA for $n \geq 3$ [16, 40], as do $GL(n, \mathbf{Z})$ and mapping class groups. Finite-index subgroups of $GL(n, \mathbf{Z})$ also have property FA; this follows from the fact that $GL(n, \mathbf{Z})$ has property (T), and property (T) descends to finite-index subgroups.

McCool [96] has found a finite-index subgroup K of $Out(F_3)$ which is residually torsion-free nilpotent. In particular, there is a map of K onto \mathbf{Z} , so that K has a non-trivial action on a tree; in particular, $Out(F_3)$ does not have property (T). McCool's result is very specific to the case $n = 3$, and it remains an open question whether $Out(F_n)$ has property (T), or whether subgroups of finite index in $Out(F_n)$ have property FA, for $n > 3$.

In [85], Lubotsky and Pak note the connection between Property (T) and expanding graphs, and remark that if $Aut(F_n)$ has Property (T), then one can make Cayley graphs on a sequence of symmetric groups into a family of expanders, thus solving an open problem in the field.

§3. Questions

Many interesting questions about free groups and their automorphisms can be found in the Magnus archive, at <http://www.grouptheory.org/>. Below I have collected the questions which were discussed in the above text, as well as some other questions which interest me.

Cohomology

1. Very few invariants for $Out(F_n)$ and $Aut(F_n)$ have been computed explicitly; extend the list, e.g. of rational cohomology groups.
2. Is any of the cohomology of mapping class groups or $GL(n, Z)$ detected by the natural maps to and from $Out(F_n)$?
3. The virtual cohomological dimension of $Out(F_n)$ is $2n - 3$. In all cases computed so far, the top-dimensional rational homology $H_{2n-3}(Out(F_n); \mathbf{Q}) = 0$. A conjecture of Morita implies that in fact the top-dimensional homology vanishes for all n . Is this the case?
4. Morita constructs rational homology classes in $H_{4k}(Out(F_{2k+2}))$ for all k , and shows that for $k = 1$ these classes are non-trivial. Are they non-trivial for all k ?
5. What is the precise stability range for the homology of $Aut(F_n)$, with integral or rational coefficients? Of $Out(F_n)$? Of the map $Aut(F_n) \rightarrow Out(F_n)$? In particular, is the stability range for the latter map linear? Ferenc Gerlits has shown that $H_7(Aut(F_5); \mathbf{Q}) = \mathbf{Q}$ while $H_7(Out(F_5); \mathbf{Q}) = 0$.

Other invariants

6. Is the Euler characteristic of $Out(F_n)$ negative for all n ? Or even non-zero? What is its growth rate?
7. Find better bounds for the Dehn function for $Out(F_n)$, $n \geq 4$; in particular, is it polynomial? Same question for higher-dimensional isoperimetric functions.
8. Arithmetic groups have infinite index in their commensurators. Ivanov shows mapping class groups are not arithmetic by showing that the commensurator of the mapping class group is just the extended mapping class group, in which the mapping class group has index 2. What is the commensurator of $Out(F_n)$ or $Aut(F_n)$?

Properties

9. Is $Out(F_3)$ linear?
10. Is $Aut(F_2)$ biautomatic?
11. Is $Aut(F_n)$ or $Out(F_n)$ automatic for $n > 3$? Is either of them asynchronously automatic, for $n \geq 3$?

12. Is $Out(F_n)$ quasi-isometrically rigid?
13. Can a map from a uniform lattice Γ in a semi-simple higher rank Lie group have infinite image in $Out(F_n)$?
14. Does $Out(F_n)$ have Kazhdan's Property T for any value of $n > 3$?
15. Do finite index subgroups of $Out(F_n)$ have Serre's property FA? i.e., can a subgroup of finite index in $Out(F_n)$, $n > 3$ act on a tree with no fixed point?
16. Is $Out(F_n)$ virtually residually torsion-free nilpotent, for $n > 3$?
17. If H is a characteristic subgroup of F_n (e.g. a verbal subgroup) of finite index, the kernel of the induced map $Aut(F_n) \rightarrow Aut(F_n/H)$ is called a *congruence subgroup*. With this definition, does $Aut(F_n)$ have the congruence subgroup property?

Outer space

18. Is the spine K_n of Reduced Outer space the minimal contractible invariant subspace of Outer space?
19. Is the set of train tracks for an irreducible automorphism contractible?
20. Bestvina and Feighn proved that $Out(F_n)$ is $(2n - 5)$ -connected at infinity, which implies that the spine K_n of Outer space is $(2n - 5)$ -connected at infinity. Find a direct proof that K_n is $(2n - 5)$ -connected at infinity.
21. What is the homotopy type of the Borel-Serre boundary of Outer space?

Miscellaneous

22. (Magnus) Is the kernel of the map from $Out(F_n)$ to $GL(n, \mathbf{Z})$ finitely presented, for $n > 3$? In general, what are the homological finiteness properties of the kernel?
23. (Casson) Let h be an automorphism of F_n . Is there a finite index subgroup G of F_n , stabilized by some power of h , such that the induced map on $H_1(K)$ has an eigenvalue which is a root of unity? (after possibly passing to a further power of h)? This is related to the covering conjecture for 3-manifolds.
24. (Burillo) It is a surprising fact that $SL(2, \mathbf{Z})$ is not quasi-convex in $SL(3, \mathbf{Z})$. Is $Aut(F_n)$ quasi-convex in $Aut(F_{n+1})$?
25. (Levitt) Which automorphisms of F_n preserve an order? Does $Out(F_n)$ have orderable subgroups of finite index?
26. (Mosher) Do there exist non-free subgroups of $Out(F_n)$ all of whose elements are irreducible of exponential growth?
27. (Mosher) Let Q be a subgroup of $Out(F_n)$ all of whose elements are irreducible with exponential growth. Is the semidirect product $F_n \rtimes Q$ word hyperbolic?

28. (Stallings) Given a reasonable measure of the complexity of an automorphism of F_n , compute a bound $f_n(k)$ such that if an automorphism α has a cyclic fixed subgroup $Fix(\alpha)$ and has complexity at most k , then the generator of $Fix(\alpha)$ has word length at most $f_n(k)$.

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