WORKSHEET: RADIUS OF CONVERGENCE

MATH 1220

Theorem: Let

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

be a power series. There is an $0 \le R \le \infty$ such that the series converges absolutely and uniformly for $0 \le |x-c| < R$ and diverges for $|x-c| > R$. Furthermore, the power series converges uniformly on the interval $|x-c| \le \rho$ for every $0 \le \rho < R$, and the limit function is continuous in $|x-c| < R$.

Lemma: If the series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges at some $x_0 \neq c$ then it converges absolutely for x such that $|x-c| < |x_0 - c|$.

Proof: Complete the details in the following steps.

- Since $\sum_{n=0}^{\infty} a_n(x_0-c)^n$ converges there is N such that for all $n > N$, $|a_n(x_0-c)^n| < 1$.
- If $0 < |x-c| < |x_0-c|$ set $r = \frac{|x-c|}{|x_0-c|}$. The series $\sum_{n=0}^{\infty} r^n$ converges.
- $\sum_{n=N+1}^{\infty} |a_n(x-c)^n| = \sum_{n=N+1}^{\infty} |a_n(x-c)^n| \frac{|(x_0-c)^n|}{|(x_0-c)^n|} \le \sum_{n=N+1}^{\infty} r^n$.
- If $0 < |x-c| < |x_0-c|$, then $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges absolutely at x.

Proof of the Theorem: Let

$$R = \sup\left\{ |x-c| \ : \ \sum_{n=0}^{\infty} a_n(x-c)^n \text{ converges} \right\}.$$

If $R = 0$ then the series converges only for $x = c$. If $R > 0$ and $|x-c| < R$ then (justify!) there is x_0 such that $|x-c| < |x_0-c| < R$; therefore, by the Lemma, the series converges absolutely at x. The definition of R implies that the series diverges for x with $|x-c| > R$ - justify.

Finally, let $0 \le \rho < R$. Choose $\sigma > 0$ such that $\rho < \sigma < R$. Justify the following steps.

- $\sum |a_n \sigma_n|$ converges, and for n big enough $|a_n \sigma^n| < 1$.
- Set $r = \frac{\rho}{\sigma}$. For n big enough and $|x-c| \le \rho$,

$$|a_n(x-c)^n| = |a_n \sigma^n| \left| \frac{x-c}{\sigma} \right|^n \le r^n.$$

- Apply the M-test to get that the series converges uniformly on $|x-c| \le \rho$.
- Conclude that the limit function is continuous on the interval $|x-c| \le \rho$.
- Since this holds for every $0 \le \rho < R$, the limit function is continuous in $|x-c| < R$.

Theorem: Suppose that $a_n \neq 0$ for all sufficiently large n and the limit

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists. Then the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has radius of convergence R where

- $R = \frac{1}{\rho}$ if $\rho > 0$, and
- $R = \infty$ if $\rho = 0$.

Proof: At a fixed x, consider the limit of ratios

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}\right| = |x-c|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-c|\rho.$$

If $\rho > 0$, use the ratio test to justify that the power series converges at x if $|x-c| < \frac{1}{\rho}$ and diverges if $|x-c| > \frac{1}{\rho}$. Check what happens if $\rho = 0$.