(Very) basic introduction to special relativity

Math 1600 Totally Awesome Mathematics, Spring 2016

Raul Gomez  Cornell University
Outline

1 Newton’s reference frames
   1.1 Newton’s three laws
   1.2 Systems of particles
   1.3 Absolute vs. relative point of view
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2 Classical Electromagnetism
   2.1 Maxwell equations
   2.2 Electromagnetic waves
   2.3 The Michelson-Morley experiment
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3 A little of linear algebra
   3.1 Matrices and the dot product
   3.2 The special orthogonal group

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   2.2 Electromagnetic waves
   2.3 The Michelson-Morley experiment

3 A little of linear algebra
   3.1 Matrices and the dot product
   3.2 The special orthogonal group

4 Special relativity
   4.1 Space-time coordinates
   4.2 The Minkowski metric
   4.3 Special relativity
Newton’s reference frames

Figure: Isaac Newton
Newton’s three laws

1. Law of inertia An object either is at rest or moves at a constant velocity, unless acted upon by an external force.
Newton’s three laws

1. **Law of inertia**  An object either is at rest or moves at a constant velocity, unless acted upon by an external force.

2. **Law of motion**  The acceleration of a body is directly proportional to the force acting on the body, and inversely proportional to its mass. In mathematical terms,

   \[ F = ma. \]
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\[ \mathbf{F} = m \mathbf{a} \]

3. **Law of action-reaction**  For every action there is an equal and opposite reaction.
About Newton’s first law

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Any such a frame is called an inertial frame.

Although this statement may seem obvious, it is not clear at all how to choose such a reference frame in the universe.
Example

If we use a fixed point on earth to define a reference frame, then we would obtain an example of a non inertial frame.
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If we use a fixed point on earth to define a reference frame, then we would obtain an example of a non inertial frame. In this frame the centrifugal force and the Coriolis force (which is responsible for the formation of hurricanes) violate Newton’s first law.
Now assume that we have a system of $N$ particles with position $\mathbf{r}_i$ and mass $m_i$. 

Newton's second law reads:

$$F_i = m_i a_i.$$ 

Decompose $F_i$ as

$$F_i = \sum_{j \neq i} F_{ij} + F_{\text{ext}i},$$

where $F_{ij}$ is the force on the $i$-th particle due to the $j$-th particle.
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Summing over all the $i$’s we get

$$\sum_i \mathbf{F}_i = \sum_{i,j,j \neq i} \mathbf{F}_{ij} + \sum_i \mathbf{F}_{i}^{\text{ext}} = \sum_{i<j} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) + \sum_i \mathbf{F}_{i}^{\text{ext}}.$$
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$$F_i = \sum_{j \neq i} F_{ij} + F_{i}^{\text{ext}},$$

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Suming over all the $i$’s we get

$$\sum_{i} F_{i} = \sum_{i,j,j \neq i} F_{ij} + \sum_{i} F_{i}^{\text{ext}}$$

$$= \sum_{i<j} (F_{ij} + F_{ji}) + \sum_{i} F_{i}^{\text{ext}}.$$

But according to Newton’s third law, $F_{ij} = -F_{ji}$, hence

$$\sum_{i} F_{i} = \sum_{i} F_{i}^{\text{ext}} =: F_{\text{ext}},$$

where $F_{\text{ext}}$ is the total exterior force acting on the system.
We define the total mass of the system to be

\[ M = \sum_i m_i, \]

and the center of mass to be

\[ \mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{M}. \]
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Then, if \( \mathbf{A} \) is the second derivative of \( \mathbf{R} \) with respect to time, we have that

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\[ \mathbf{F}^{\text{ext}} = MA. \]

In a system where there are no external forces (like the universe) this equation says that fixing the origin at the center of mass gives you an **inertial** reference frame.
Example

Let

\[ \mathbf{r}_1 = (0,0) \quad \mathbf{r}_2 = (3,6) \quad \mathbf{r}_3 = (8,2), \]

and assume that

\[ \mathbf{F}_{12} = (1,2) \quad \mathbf{F}_{13} = (2,1/2) \quad \mathbf{F}_{23} = (0,0). \]

Then we obtain the following picture:
Absolute vs. relative point of view

Figure: Gottfried Leibniz
Classical Electromagnetism

Figure: James Clerk Maxwell
Maxwell equations (1862)

Let \( \mathbf{E} \), \( \mathbf{B} \) and \( \mathbf{J} \) be the electric field, magnetic field and current density, respectively, and let \( \rho \) be a charge distribution in \( \mathbb{R}^3 \).

1. Gauss law.

\[
\nabla \cdot \mathbf{E} = \rho / \varepsilon_0.
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Maxwell equations (1862)

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2. **Gauss law for magnetism.**

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\nabla \cdot \mathbf{B} = 0.
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3. **Fadaray Law.**

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.
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Maxwell equations (1862)

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$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$  

4. Ampère circulation law (with Maxwell correction.)

$$\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right),$$  

where $\varepsilon_0$ and $\mu_0$ are the electric and magnetic constants.
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Let \( E, B \) and \( J \) be the electric field, magnetic field and current density, respectively, and let \( \rho \) be a charge distribution in \( \mathbb{R}^3 \).

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\nabla \times B = \mu_0 \left( J + \varepsilon_0 \frac{\partial E}{\partial t} \right),
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More precisely, if \( \rho = 0 \), then Maxwell equations are equivalent to the system of equations:

\[
\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0
\]

and

\[
\frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 B}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = 0,
\]

plus a condition relating \( E \) and \( B \).
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and

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plus a condition relating $E$ and $B$.

Here

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

is the speed of light.
Electromagnetic waves

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Based on this observation, Maxwell predicted that light was just an example of an electromagnetic wave.
Can electromagnetic waves propagate in the vacuum?

What’s wrong with the following video?
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1. On earth, waves always propagate through a medium, they can’t exist in isolation.

2. Maxwell calculated the speed of the electromagnetic waves, but this was the speed with respect to what exactly?
The aether hypothesis

Trying to kill two birds in one stone, they proposed the existence of an essentially indetectible substance called luminiferous aether. This was supposed to be the medium through which electromagnetic waves propagated. The speed calculated by Maxwell was then the speed of light with respect to this medium.
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Figure: Aether
The Michelson Morley experiment

So in 1887, Albert Michelson and Edward Morley designed an experiment to measure the velocity of the earth with respect to the surrounding aether.
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So in 1887, Albert Michelson and Edward Morley designed an experiment to measure the velocity of the earth with respect to the surrounding aether.

The result however, was totally unexpected. The speed of light was the same in every direction!
Figure: Albert Einstein
The dot product

We will now introduce a little bit of linear algebra in the plane. If \( v_1 = (x_1, y_1) \), \( v_2 = (x_2, y_2) \) are two vectors on the plane, then we define

\[
v_1 + v_2 := (x_1 + x_2, y_1 + y_2),
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and given \( c \in \mathbb{R} \), we define

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  cv_1 := (cx_1, cy_1).
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We also define the dot product between $v_1$ and $v_2$ to be

$$v_1 \cdot v_2 = x_1 x_2 + y_1 y_2,$$
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and given \( c \in \mathbb{R} \), we define

\[
c \mathbf{v}_1 := (cx_1, cy_1).
\]

We also define the **dot product** between \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) to be

\[
\mathbf{v}_1 \cdot \mathbf{v}_2 = x_1x_2 + y_1y_2,
\]

and the **norm** of a vector to be

\[
\|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}}.
\]

Observe that this is just the distance between the origin and \( \mathbf{v} \).
Lemma

If \( \mathbf{v} \) and \( \mathbf{w} \) are two vectors in \( \mathbb{R}^2 \), then

\[
\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| ||\mathbf{w}|| \cos \theta,
\]

where \( \theta \) is the angle between \( \mathbf{v} \) and \( \mathbf{w} \).
Matrices

**Definition**

A $2 \times 2$ matrix, is an array of four real numbers of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
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Given a matrix $A$ as above, and a vector $v = (x, y)$, we define

$$Av = (ax + by, cx + dy).$$
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The equation above is normally written in the following way:

$$Av = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$
Definition

Let

\[ \text{SO}(2, \mathbb{R}) = \left\{ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \bigg| \theta \in \mathbb{R} \right\}. \]

\( \text{SO}(2, \mathbb{R}) \) is called the \textbf{special orthogonal} group, and it consists of all rotations in the plane.
Definition

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$$SO(2, \mathbb{R}) = \left\{ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \mid \theta \in \mathbb{R} \right\}.$$  

$SO(2, \mathbb{R})$ is called the special orthogonal group, and it consists of all rotations in the plane.

It actually consists of all the transformations in the plane that preserve distance and orientation.
Lemma

If \( v_1, v_2 \in \mathbb{R}^2 \) and \( k \in \text{SO}(2, \mathbb{R}) \), then

\[
k v_1 \cdot (k v_2) = v_1 \cdot v_2.
\]
Lemma

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\]

Lemma

Given \( \theta \in \mathbb{R} \), set

\[
k(\theta) := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.
\]

Then, for all \( \theta_1, \theta_2 \in \mathbb{R} \) and all \( v \in \mathbb{R}^2 \),

\[
k(\theta_1)(k(\theta_2)v) = k(\theta_1 + \theta_2)v.
\]
Space-time coordinates

To simplify calculations, from now on we will only consider universes with one spatial dimension.
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That is, we will consider events in the universe by a space coordinate and a time coordinate.
**4.1 Space-time coordinates**

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That is, we will consider events in the universe by a space coordinate and a time coordinate.

Observe that if \( r(t) \) represents the position of a particle at time \( t \), then we can describe its position as a point in \( \mathbb{R}^2 \), \((t, r(t))\).
The Minkowski metric

Figure: Hermann Minkowski
The Minkowski metric

Is it possible to define a concept of distance between different events in the universe?
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But if we ask about the distance between this moment and the superbowl, we only need to numbers: Santa Clara is about 2810 miles from here, and the superbowl happened 30 days ago.
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But if we ask about the distance between this moment and the superbowl, we only need to numbers: Santa Clara is about 2810 miles from here, and the superbowl happened 30 days ago.

Can we combine this 2 numbers to get a notion of “distance” between this two events?
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Unfortunately this is not very useful because the laws of physics are not invariant under the action of $\text{SO}(2, \mathbb{R})$.

For example, if we write Maxwell equations and then we transform everything using an element of $\text{SO}(2, \mathbb{R})$ we get an equation that looks very different.
This is clear if we look at the wave equation in one dimension:

\[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0. \]
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Looking at this equation suggest a different way of calculating the "distance" between 2 events.
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Looking at this equation suggest a different way of calculating the “distance” between 2 events.

**Definition**

Let \( v_1 = (t_1, x_1) \) and \( v_2 = (t_2, x_2) \) be two vectors in \( \mathbb{R}^2 \). We define the **Minkowski product** of \( v_1 \) and \( v_2 \) by

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v_1 \circ v_2 = t_1 t_2 - x_1 x_2.
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Using this inner product, we can define a new "norm" (or rather its square) by

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N(\mathbf{v})^2 = \mathbf{v} \circ \mathbf{v},
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**Definition**

Let $v_1 = (t_1, x_1)$ and $v_2 = (t_2, x_2)$ be two vectors in $\mathbb{R}^2$. We define the **Minkowski product** of $v_1$ and $v_2$ by

$$v_1 \circ v_2 = t_1 t_2 - x_1 x_2.$$ 

Using this inner product, we can define a new “norm” (or rather its square) by

$$N(v)^2 = v \circ v,$$

and using this norm we can define the Minkowski “metric”

$$d(v_1, v_2)^2 = N(v_1 - v_2)^2.$$
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The name “metric” here is a little misleading because the number I’m getting on the right may very well be negative, in which case, taking the square root is not really well defined.
and using this norm we can define the Minkowski “metric”

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The name “metric” here is a little misleading because the number I’m getting on the right may very well be negative, in which case, taking the square root is not really well defined.

However the Minkowski product of a vector in \( \mathbb{R}^2 \) with itself is well defined and is a very useful quantity as we will see soon.
The other special orthogonal group

**Definition**

Let

\[ \text{SO}(1, 1) = \left\{ \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix} \mid x \in \mathbb{R} \right\}. \]
The other special orthogonal group

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$$SO(1, 1)_o = \left\{ \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix} \mid x \in \mathbb{R} \right\}.$$

We will show that this group is related to the Minkowski product in pretty much the same way as $SO(2, \mathbb{R})$ was related to the usual dot product.
Lemma

If $v_1, v_2 \in \mathbb{R}^2$ and $g \in SO(1, 1)$, then

$$gv_1 \circ (gv_2) = v_1 \circ v_2.$$
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If $v_1, v_2 \in \mathbb{R}^2$ and $g \in SO(1, 1) \circ$, then

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Lemma

Given $x \in \mathbb{R}$, set

$$g(x) := \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix}. $$

Then, for all $x_1, x_2 \in \mathbb{R}$ and all $v \in \mathbb{R}^2$,

$$g(x_1)(g(x_2)v) = g(x_1 + x_2)v.$$

The point of the Minkowski inner product is that Maxwell equations (and in general the equations of physics) are preserved under the action of $\text{SO}(1, 1)$. 
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That is, I can use \( \text{SO}(1,1) \) to change my reference frame and still get the same physics laws.

As we will see shortly, this is extremely useful.
Let’s start by considering the points in $\mathbb{R}^2$ such that $v \circ v = 0$.

This are called the light rays, precisely because they describe the trajectory of light coming up from the origin.
The rest of the plane gets divided into two regions, the \textit{time-like} directions:

\[
\{ v \in \mathbb{R}^2 \mid v \circ v > 0 \},
\]

and the \textit{space-like} directions:

\[
\{ v \in \mathbb{R}^2 \mid v \circ v < 0 \}.
\]
Observe now that the set
\[ \{ v \in \mathbb{R}^2 \mid v \circ v = 1 \}, \]
describes an hyperbola. This are the points that, in a sense, are at “distance” 1 from the origin.
Now we will consider two particles, $r_1$ and $r_2$ traveling at constant speed from the origin and whose graph in $\mathbb{R}^2$ is given by the parametric equations: $t \mapsto (t, 0)$, $t \mapsto (t \cosh \alpha, t \sinh \alpha)$. 
If we transform this picture using the element $g(-\alpha)$ we obtain:

Observe that this means that for both observers their relative velocities are the same, and the speed of light is the same with respect to both observers!
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