Definition 0.1 (Infinite Sequences) *An infinite sequence of numbers is a function whose domain is the set of positive integers.*

Ex 1. *Write out the first few terms of the sequences* $a_1 = 1,\ a_{n+1} = a_n + \frac{1}{2^n}$.

sol) $a_1 = 1,\ a_2 = a_1 + \frac{1}{2} = \frac{3}{2},\ a_3 = a_2 + \frac{1}{4} = \frac{6+1}{4} = \frac{7}{4}$

Ex 2. *Find a formula for the nth term of the sequence.*

a. *The sequence* $1,\ -4,\ 9,\ -16,\ 25, ...$
sol) $a_n = (-1)^{1+n}n^2$

b. *The sequence* $1,\ 0,\ 1,\ 0,\ 1, ...$
sol) $a_n = \frac{1+(-1)^{1+n}}{2}$

Theorem 0.2 *Suppose that* $f(x)$ *is a function defined for all* $x \geq n_0$ *and that* $\{a_n\}$ *is a sequence of real numbers such that* $a_n = f(n)$ *for* $n \geq n_0$*. Then*

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{x\to\infty} a_n = L$$

The significance of Theorem 0.2 is to enable us to use L'Hôpital's Rule to find the limits of some sequences.

Ex 3. *Find* $\lim_{n\to\infty} a_n$ *when* $a_n = \frac{1-5n^4}{n^4+8n^3}$.

sol) Let $f(n) = a_n$ for $n \geq 1$. Then

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{1-5x^4}{x^4+8x^3} \\
&= \lim_{x\to\infty} \frac{-20x^3}{4x^3+24x^2} && (1) \\
&= \lim_{x\to\infty} \frac{-60x^2}{12x^2+48x} && (2) \\
&= \lim_{x\to\infty} \frac{-120x}{24x+48} && (3) \\
&= \lim_{x\to\infty} \frac{-120}{24} && (4) \\
&= -5, && (5)
\end{aligned}$$

where (1), (2), (3), and (4) are from L'Hôpital's Rule. By Theorem 0.2, (5) implies $\lim_{n\to\infty} a_n = -5$.

Ex 4. *Find* $\lim_{n\to\infty} a_n$ *when* $a_n = \frac{(\ln n)^{200}}{n}$.

sol) Let $f(n) = a_n$ for $n \geq 1$. Then

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{(\ln x)^{200}}{x} \\
&= \lim_{x\to\infty} \frac{200(\ln x)^{199}\frac{1}{x}}{1} && (6) \\
&= \lim_{x\to\infty} \frac{200(\ln x)^{199}}{x} \\
&= ... && (7) \\
&= \lim_{x\to\infty} \frac{200!(\ln x)}{x} && (8) \\
&= 0, && (9)
\end{aligned}$$

where (6), (7), and (8) are from L'Hôpital's Rule. By Theorem 0.2, (5) implies $\lim_{n\to\infty} a_n = 0$.

Definition 0.3 (Geometric Series) *a. If* $|r| < 1$, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

b. If $|r| \geq 1$, $\sum_{n=1}^{\infty} ar^{n-1} = \infty$. *Diverges.*

Ex 5. *Find* $\sum_{n=1}^{\infty} \frac{7}{4^n}$.

sol) Let $r = \frac{1}{4}$. Since this is a sum of the geometric series with $r = \frac{1}{4}$, $\sum_{n=1}^{\infty} \frac{7}{4^n} = \frac{\frac{7}{4}}{1-\frac{1}{4}} = \frac{7}{3}$.

Ex 6. *Find* $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n} \right)$.

sol) Let $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{5}$. Then this is a sum of two geometric series with $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{5}$, $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n} \right) = \frac{1}{1-\frac{1}{2}} + \frac{1}{1-(-\frac{1}{5})} = 2 + \frac{5}{6} = \frac{17}{6}$.

Theorem 0.4 *a. If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $a_n \to 0$. *Note that the converse is not true, i.e.* $a_n \to 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ *converges. For example,* $a_n = \frac{1}{n}$.

b. If a_n *fails to exist or is difference from zero, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Ex 7. *Does* $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^n$ *converge or diverge? If a series converge, find its sum.*

sol) Let $a_n = \left(\frac{1}{\sqrt{2}} \right)^n$. Notice that $\left(\frac{1}{\sqrt{2}} \right)^n \to 0$, and Theorem 0.4 doesn't tell us if $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^n$ converges. However, this is a geometric series with $r = \frac{1}{\sqrt{2}}$. Therefore $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^n = \frac{1}{1-\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1} = \sqrt{2}(\sqrt{2}+1) = 2 + \sqrt{2}$.

Ex 8. *Does* $\sum_{n=0}^{\infty} (-1)^{n+1} n$ *converge or diverge? If a series converge, find its sum.*

sol) $\lim_{n\to\infty} (-1)^{n+1} n$ fails to exist because it oscillates. By Theorem 0.4, $\sum_{n=0}^{\infty} (-1)^{n+1} n$ diverges.