Theorem 0.1 (The nth Term Test) *If a_n fails to exist or is different from zero, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Theorem 0.2 (The Integral Test) *Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where $f(x)$ is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.*

Theorem 0.3 (The Direct Comparison Test) *Let a_n be a series with no negative terms.*

a. *$\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n > N$, for some integer N.*

b. *$\sum a_n$ diverges if there is a divergent series $\sum d_n$ with $a_n \geq d_n$ for all $n > N$, for some integer N.*

Theorem 0.4 (The Limit Comparison Test) *Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).*

a. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.*

b. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.*

c. *If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

Theorem 0.5 (The Ratio Test) *Let $\sum a_n$ be a series with positive terms and suppose that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

a. *the series converges if $\rho < 1$,*

b. *the series diverges if $\rho > 1$ or ρ is infinite,*

c. *the test is inconclusive if $\rho = 1$.*

Theorem 0.6 (The Root Test) *Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

Then

a. *the series converges if $\rho < 1$,*

b. *the series diverges if $\rho > 1$,*

c. *the test is inconclusive if $\rho = 1$.*

Which of the series in the following converge, and which diverge?

1. $\sum_{n=1}^{\infty} \frac{1}{10^n}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{-2}{n+1}$

sol) $\displaystyle\sum_{n=0}^{\infty} \frac{-2}{n+1} = -\sum_{n=0}^{\infty} \frac{2}{n+1}$. Use the Integral Test. Let $f(x) = \frac{2}{x+1}$ and choose $N = 0$. The function $f(x) = \frac{2}{x+1}$ is continuous, positive, and decreasing. Thus $\int_0^\infty \frac{2}{x+1} dx = \lim_{b\to\infty} 2[\ln|x+1|]_0^b = \infty - 2 = \infty$. By the Integral Test, $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n+1}$ diverges. $\Rightarrow \displaystyle\sum_{n=0}^{\infty} \frac{-2}{n+1}$ diverges.

3. $\displaystyle\sum_{n=3}^{\infty} \frac{\frac{1}{n}}{(\ln n)\sqrt{(\ln n)^2 - 1}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$

sol) Use the Integral Test. Let $f(x) = \frac{8\tan^{-1} x}{1+x^2}$, which is a continuous, positive, and decreasing function for $x \geq 1$ and $N = 1$. Use the substitution $x = \tan u$ to get $\int_1^\infty \frac{8\tan^{-1} x}{1+x^2} dx = \int_{\pi/4}^{\pi/2} \frac{8u}{\sec^2 u} \sec^2 u du = 4u^2|_{\pi/4}^{\pi/2} = 4(\pi^2/4 - \pi^2/16) = \frac{3\pi^2}{4}$. Therefore by the Integral Test, $\displaystyle\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$ converges.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}}$

sol) Use the Limit Comparison Test. Let $a_n = \frac{1}{\sqrt{n^3+2}}$ and $b_n = \frac{1}{n^{3/2}}$. $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}}{(n^3+2)^{3/2}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{2}{n^3}}} = 1$. Since $\sum \frac{1}{n^{3/2}}$ converges, $\sum \frac{1}{\sqrt{n^3+2}}$ also converges by the Limit Comparison Test (Part a).

7. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

sol) Use the Limit Comparison Test. Let $a_n = \frac{1}{n\sqrt{n^2-1}}$ and $b_n = \frac{1}{n^2}$. $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2-1}} = \lim_{n\to\infty} \sqrt{\frac{n^4}{n^4-n^2}} = \lim_{n\to\infty} \sqrt{\frac{1}{1-\frac{1}{n^2}}} = 1$. Since $\sum \frac{1}{n^2}$ converges, $\sum \frac{1}{n\sqrt{n^2-1}}$ also converges by the Limit Comparison Test (Part a).

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1-n}{n \cdot 2^n}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$

sol) By the Ration Test, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{\sqrt{2}}}{2^{n+1}} \cdot \frac{2^n}{n^{\sqrt{2}}} = \frac{1}{2} < 1$ converges.

11. $\displaystyle\sum_{n=1}^{\infty} n!e^{-n}$

12. $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^n$

sol) By the nth Term Test, $\lim_{n\to\infty} \left(1 - \frac{3}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1-\frac{3}{n}\right)} = \lim_{n\to\infty} e^{\frac{\ln\left(1-\frac{3}{n}\right)}{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{\frac{\frac{3}{n^2}}{1-\frac{3}{n}}}{-\frac{1}{n^2}}} =$ $e^{-3} \neq 0$. Hence $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^n$ diverges.

13. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)!}{3!n!3^n}$