1. *Which of the following functions have inverses?*

   a. $f(x) = (x-1)^3$ with domain $(-\infty, \infty)$.
   $f(x)$ is invertible because it is a strictly increasing function (that is, $f(a) > f(b)$ whenever $a > b$).

   b. $f(x) = \frac{1}{2+\sin x}$ with domain $[-\pi/2, \pi/2]$.
   $f(x)$ is invertible as it is one-to-one: if $f(a) = f(b)$ then $\frac{1}{2+\sin a} = \frac{1}{2+\sin b}$ and so $\sin a = \sin b$ and so $a = b$, as $\sin x$ is one-to-one on the domain $[-\pi/2, \pi/2]$.

   c. $f(x) = \sec^2 x$ with domain $(-\pi/2, \pi/2)$.
   $f(x)$ is not invertible as it is not one-to-one: $f(x) = f(-x)$ for all $x \neq 0$.

2. a. *Show that the function $f(x) = x^3 + 3x$ is one-to-one and hence has an inverse $f^{-1}(x)$.*

   b. *What are the domain and the range of $f^{-1}(x)$?*

   c. *Find $\frac{d}{dx} f^{-1}(x)$ at $x = 4$.*

3. *Let $f(x) = \sin^{-1}(\tan x)$.*

   a. *What is the natural domain for $f(x)$ so that the inverse function $f^{-1}(x)$ exists?*

   Step 1. The function $f(x)$ must be one-to-one so that the inverse function $f^{-1}(x)$ exists. The function $f(x)$ is one-to-one if and only if the functions $\sin^{-1} x$ and $\tan x$ are one-to-one.

   Step 2. The domain $\mathcal{D}_1$ of $\tan x$ is $(-\infty, \infty)$. The range $\mathcal{R}_1$ is $(-\infty, \infty)$. However we must restrict the function to be one-to-one so that the inverse will exist. Thus, we set the domain to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the range to $(-\infty, \infty)$. Similarly, the domain $\mathcal{D}_2$ of $\sin^{-1} x$ is $[-1, 1]$. The range $\mathcal{R}_2$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The function $\sin^{-1} x$ is one-to-one in the domain $\mathcal{D}_2$ and the range $\mathcal{R}_2$, and so the inverse exists.

   Step 3. The range $\mathcal{R}_1$ of $\tan x$ must agree with the domain $\mathcal{D}_2$ of $\sin^{-1} x$, i.e., $\mathcal{R}_1 = \mathcal{D}_2 = [-1, 1]$. To achieve this we will restrict the domain $\mathcal{D}_1$ of $\tan x$ to $[-\frac{\pi}{4}, \frac{\pi}{4}]$. Thus, the domain of $f(x)$ is $[-\frac{\pi}{4}, \frac{\pi}{4}]$ and the range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

   b. *Calculate the inverse function $f^{-1}(x)$ and specify its domain.*
   The inverse function $f^{-1}(x) = \tan^{-1}(\sin x)$. Pulling directly from the range above, the domain is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

4. *Evaluate* $\int \frac{dx}{5x\sqrt{\ln(3x)}}$.

5. *Evaluate* $\frac{\ln x}{x(1+\ln x)} dx$.

6. *Evaluate* $\int_0^{\sqrt{\ln \pi}} 2xe^{x^2} \cos(e^{x^2}) dx$.