Let $f(x)$ be a continuous function on $[a,\ b]$. Then by the fundamental theorem of calculus, $F(x) = \int_a^x f(t)dt$ is continuous on $[a,\ b]$, is differentiable on $[a,\ b]$, and

$$F'(x) = \frac{dF(x)}{dx} = \frac{d\int_a^x f(t)dt}{dx} = f(x). \tag{1}$$

Now consider $g(x)$ instead of x as an upper bound of the integral above. Then to find $F'(g(x))$, we need to use the chain rule to get

$$\begin{aligned} F'(g(x)) &= \frac{dF(g(x))}{dx} && (2)\\ &= \frac{dF(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx} && (3)\\ &= \frac{d\int_a^{g(x)} f(t)dt}{dg(x)} \cdot g'(x) && (4)\\ &= f(g(x)) \cdot g'(x), && (5) \end{aligned}$$

where (3) is from the chain rule for the mapping $x \to g(x) \to F(g(x))$, and (5) from (1) with $g(x)$ instead of x, i.e. the fundamental theorem of calculus.

1 *Find the derivative* $\frac{d}{dt}\int_0^{t^4} \sqrt{u}du$ *by evaluating the integral and differentiating the result,*

$\frac{d}{dt}\int_0^{t^4} \sqrt{u}du = \frac{d}{dt}\left(\frac{2}{3}u^{3/2}|_0^{t^4}\right) = \frac{d}{dt}\left(\frac{2}{3}t^6\right) = 4t^5$

2 *Find the derivative* $\frac{d}{dt}\int_0^{t^4} \sqrt{u}du$ *by differentiating the integral directly.*

Since the upper bound is not simply x, We use the fundamental theorem of calculus combining with the chain rule above. Let $f(u) = \sqrt{u}$ and $g(t) = t^4$. Then

$$\begin{aligned} \frac{d}{dt}\int_0^{t^4} \sqrt{u}du &= \frac{d}{dt}\int_0^{g(t)} f(u)du && (6)\\ &= f(g(t)) \cdot g'(t) && (7)\\ &= \sqrt{t^4} \cdot \frac{d(t^4)}{dt} && (8)\\ &= t^2 \cdot 4t^3 && (9)\\ &= 4t^5 && (10) \end{aligned}$$