Practice Prelim 1 - Solutions Fall 2005

1. a) We know that $\frac{d}{dx}(\tan x) = \sec^2 x = \frac{1}{\cos^2 x}$, so it follows that $\int \sec^2(x)\,dx = \tan(x) + C$

b) By the fund. thm part 1 $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ for cont. $f(t)$,

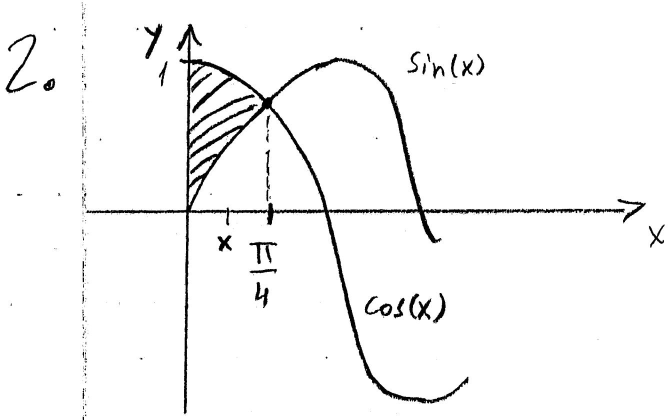
and in our case we also need to apply the chain rule:

$$\frac{d}{dt}\int_{t^3}^{1} \cos^4(x)\,dx = -\frac{d}{dt}\int_1^{t^3}\cos^4(x)\,dx = -\cos^4(t^3)\cdot\frac{d}{dt}(t^3) =$$

$$= -3t^2\cos^4(t^3).$$

c) We first note that $f(x) = \frac{x^3}{1+x^6}$ is an odd function:

$f(-x) = \frac{(-x)^3}{1+(-x)^6} = \frac{-x^3}{1+x^6} = -f(x)$, and the limits of the integration are symmetric, so we can conclude that $\int_{-2}^{2}\frac{x^3}{1+x^6}\,dx = 0$.

2. 

The first intersection point of sin(x) and cos(x) is $x = \frac{\pi}{4}$:

$\cos(x) = \sin(x) \Rightarrow \tan(x) = 1 \Rightarrow$
$x = \frac{\pi}{4} + \pi k,\ k = 0, 1, 2, \ldots$
and we take $k = 0$.

The area is given by $A = \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]\,dx = (\sin(x) + \cos(x))\Big|_0^{\frac{\pi}{4}} =$

$= \sin(\frac{\pi}{4}) + \cos(\frac{\pi}{4}) - \sin(0) - \cos(0) = 2 \cdot \frac{\sqrt{2}}{2} - 0 - 1 = \sqrt{2} - 1.$

$$\text{Area} = \sqrt{2} - 1 .$$

3. a) The area under the curve $y = \sin(x)$ between $x = 0$ and $x = \frac{\pi}{6}$ can be expressed as a definite integral $\int_0^{\frac{\pi}{6}} \sin(x)\,dx$ which is a limit of Riemann sums. To set up the Riemann sum we can take a uniform partition of $[0, \frac{\pi}{6}]$ into n segments, so $\Delta x_k = \frac{\pi}{6n}$. The right endpoint of the k-th segment $\left[\frac{\pi(k-1)}{6n}, \frac{\pi k}{6n}\right]$ is $\frac{\pi k}{6n}$, and therefore the sum is $\sum_{k=1}^{n} f(x_k)\,\Delta x_k = \sum_{k=1}^{n} \sin\left(\frac{\pi k}{6n}\right) \frac{\pi}{6n}$.

The area is given by $\lim_{n \to \infty} \sum_{k=1}^{n} \sin\left(\frac{\pi k}{6n}\right) \frac{\pi}{6n}$.

b) The idea here is to recognize that the given limit is actually a limit of Riemann sums, and therefore it is actually a definite integral of some function.

First, we rewrite $\sum_{k=1}^{n} \frac{k^7}{n^8}$ as $\sum_{k=1}^{n} \left(\frac{k}{n}\right)^7 \cdot \frac{1}{n}$, and then observe that $\frac{1}{n}$ is exactly the width of each subinterval if we consider the partition of the interval $[0, 1]$ into n equal segments.

Now if we take the function $f(x) = x^7$ on $[0,1]$ and try to set up a Riemann sum using uniform partition and the right hand rule, we get

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} = \sum_{k=1}^{n}\left(\frac{k}{n}\right)^7\cdot\frac{1}{n} = \sum_{k=1}^{n}\frac{k^7}{n^8}.$$

The limit of those Riemann sums is the definite integral of $f(x)$, so

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^7}{n^8} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^7\cdot\frac{1}{n} = \int_0^1 x^7\,dx = \frac{1}{8}x^8\Big|_0^1 = \frac{1}{8}.$$

4. a) We apply the substitution $u=\sqrt{1-x^2}$, so $du = \frac{1}{2}\cdot(-2x)\cdot(1-x^2)^{-\frac{1}{2}}dx =$

$$= \frac{-x}{\sqrt{1-x^2}}\,dx$$

and then $\displaystyle\int\frac{x\,dx}{\sqrt{1-x^2}} = \int -du = -u + C = -\sqrt{1-x^2}+C.$

<u>Note</u>: The substitution $w = 1-x^2$ would work just as well.

b) We apply the substitution $t = 2-x$, and then $dt = -dx$ and

$x^2 = (2-t)^2$. The integral becomes

$$\int x^2\sqrt{2-x}\,dx = \int (2-t)^2\sqrt{t}\cdot(-dt) = -\int (t^2-4t+4)\,t^{1/2}\,dt = -\int\left(t^{5/2}-4t^{3/2}+4t^{1/2}\right)dt$$

$$= -\frac{2}{7}t^{7/2} + 4\cdot\frac{2}{5}t^{5/2} - 4\cdot\frac{2}{3}t^{3/2} + C = -\frac{2}{7}(2-x)^{7/2} + \frac{8}{5}(2-x)^{5/2} - \frac{8}{3}(2-x)^{3/2} + C$$

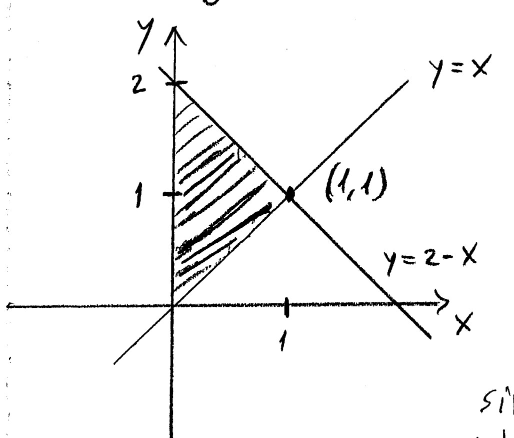
c) First we note that $\sqrt{1-\sin^2(y)} = \sqrt{\cos^2 y} = \cos y$, so the integral simplifies to
$$\int_0^1 \sqrt{1-\sin(y)}\sqrt{1-\sin^2(y)}\,dy = \int_0^1 \sqrt{1-\sin(y)}\cos(y)\,dy$$

Now apply the substitution $u = 1-\sin(y)$, so $du = -\cos(y)\,dy$

and thus $\int \sqrt{1-\sin(y)}\cos(y)\,dy = \int u^{1/2}\cdot(-du) = -\frac{2}{3}u^{3/2} + C =$
$= -\frac{2}{3}\left(1-\sin(y)\right)^{3/2} + C$. Applying the integration limits we get

$$\int_0^1 \sqrt{1-\sin(y)}\sqrt{1-\sin^2(y)}\,dy = -\frac{2}{3}\left(1-\sin(y)\right)^{3/2}\Big|_0^1 = -\frac{2}{3}\left(1-\sin(1)\right)^{3/2} + \frac{2}{3}.$$

5. a) The region we consider looks like that:



The intersection point of the two lines is (1, 1).

For this region, it is more natural to integrate w.r.t. x ("vertical strips") since this will enable us to set up only one integral.

Since the region is rotated about the y-axis, that corresponds to the "cylindrical shell" method:

$$V = \int_0^1 2\pi x\,(2-x-x)\,dx = 2\pi\int_0^1 (2x-2x^2)\,dx = 2\pi\left(x^2 - \frac{2}{3}x^3\right)\Big|_0^1 =$$
$$= 2\pi\left(1-\frac{2}{3}\right) = \frac{2\pi}{3}.$$

b) Again, we integrate with respect to x, but now the axis of rotation is the x-axis, so the rotated "vertical strips" trace out washers in space.

$$V = \int_0^1 \pi\left((2-x)^2 - x^2\right)dx = \pi \int_0^1 (x^2 - 4x + 4 - x^2)\,dx = \pi \int_0^1 (4-4x)\,dx =$$

$$= \pi\left(4x - 2x^2\right)\Big|_0^1 = \pi(4-2) = 2\pi.$$

Note: Both parts could be solved by integrating with respect to y, but in that case we would need to setup two integrals instead of one.