PRACTICE PRELIM 2, FALL 2005, MATH 191

SOLUTIONS

1. Curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ $\qquad \frac{dy}{dx} = x - \frac{1}{4x} = \frac{4x^2-1}{4x}$

The arc length element is $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{4x^2-1}{4x}\right)^2} = \sqrt{\frac{16x^2}{16x^2} + \frac{16x^4-8x^2+1}{16x^2}}$

$$= \sqrt{\frac{16x^4+8x^2+1}{16x^2}} = \sqrt{\left(\frac{4x^2+1}{4x}\right)^2}$$

$$= \frac{4x^2+1}{4|x|} = \frac{4x^2+1}{4x} \text{ if } x \geq 0.$$

(a) $L = \int_1^{e^4} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^{e^4} \frac{4x^2+1}{4x}dx = \int_1^{e^4}\left(x+\frac{1}{4x}\right)dx = \left(\frac{1}{2}x^2+\frac{1}{4}\ln|x|\right)\Big]_1^{e^4} = \boxed{\frac{1}{2}(e^8+1)}$

(b) $\frac{dy}{dx} = \frac{4x^2-1}{4x} \geq 0$ if $x>0$ and $4x^2-1\geq 0$, i.e. if $x \geq \frac{1}{2}$. For $x=1$, $y = \frac{1}{2}(1)^2 - \frac{1}{4}\ln(1) = \frac{1}{2} > 0$

Therefore the curve lies above the x-axis for $x \geq 1$ and in particular on $[1,e]$

$S = \int 2\pi(\text{radius})(\text{arc-length element}) = \int_{x=1}^{e} 2\pi\, y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^e 2\pi\left(\frac{1}{2}x^2 - \frac{1}{4}\ln x\right)\cdot\left(x+\frac{1}{4x}\right)dx$

$S = 2\pi\int_1^e \left(\frac{1}{2}x^3 + \frac{1}{8}x - \frac{1}{4}x\ln x - \frac{1}{16}\frac{\ln x}{x}\right)dx$

We have to evaluate $\int x\ln x$ and $\int \frac{\ln x}{x}dx$ by parts:

$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{x^2}{2}\cdot\frac{dx}{x} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

$u = \ln x \quad du = \frac{dx}{x}$

$dv = x\,dx \quad v = x^2/2$

$\int \frac{\ln x}{x}dx = (\ln x)^2 - \int \frac{\ln x}{x}dx \implies 2\int\frac{\ln x}{x}dx = (\ln x)^2 + C'$

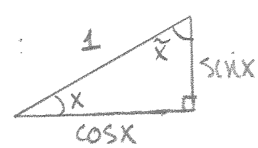
$u = \ln x \quad du = \frac{dx}{x}$

$dv = \frac{1}{x}dx \quad v = \ln x$

$\implies \int \frac{\ln x}{x}dx = \frac{1}{2}\ln^2 x + C.$

We evaluate between $x=1$ and $x=e$ to get $\boxed{S = \pi\left(\frac{e^4}{4} - \frac{9}{16}\right)}$

2. (a) $\sin^{-1}(\cos x)$ for x in $[0, \frac{\pi}{2}]$. Draw a triangle:

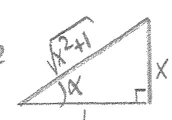


We're asking what angle has a sine equal to $\cos x$: $\frac{\cos x}{1}$.

That's the angle $\tilde{x} = \frac{\pi}{2} - x$. So $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$

Equivalently, use the identity $\cos x = \sin\left(\frac{\pi}{2}-x\right)$ for x in $[0, \frac{\pi}{2}]$

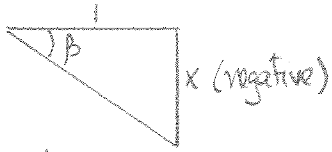
(b) $\cos(\tan^{-1}x)$. Suppose first that $x \geq 0$. Draw a triangle



$\alpha = \tan^{-1}x$.

Then $\cos(\tan^{-1}x) = \cos(\alpha) = \frac{1}{\sqrt{x^2+1}}$

For x negative: $x = -a$ $(a>0)$ and $\cos(\tan^{-1}(-a)) = \cos(-\tan^{-1}a) = \cos(\tan^{-1}a) = \frac{1}{\sqrt{a^2+1}} = \frac{1}{\sqrt{x^2+1}}$ ($\tan^{-1}$: odd f^n, cos: even f^n.)

Equivalently, draw a triangle 

$\beta = -\alpha$ but $\cos(\beta) = \cos(\alpha)$ since cosine is an even function

(c) $y = \tan^{-1}(\ln x)$ $\quad \dfrac{dy}{dx} = \dfrac{1}{1+(\ln x)^2} \cdot \dfrac{d}{dx}\ln x = \dfrac{1}{x(1+\ln^2 x)}$ $\quad \left.\dfrac{dy}{dx}\right|_{x=e} = \dfrac{1}{e(1+1^2)} = \dfrac{1}{2e}$

At $x = e$, $y = \tan^{-1}(\ln e) = \tan^{-1}(1) = \frac{\pi}{4}$. The equation of a line of slope m through (x_0, y_0) is $\dfrac{y - y_0}{x - x_0} = m$. In our case $\dfrac{y - \frac{\pi}{4}}{x - e} = \dfrac{1}{2e}$ or $\boxed{y = \frac{1}{2e}x + \left(\frac{\pi}{4} - \frac{1}{2}\right)}$

3. Let $B(t)$ denote the number of bacteria at time t (t in hours). If $B(t)$ increases exponentially with time, then $B(t) = B_0 e^{kt}$ for some positive constant k.
$B(0) = B_0 \cdot e^{k \cdot 0} = B_0$, so B_0 is the population at $t = 0$, the initial population.
The data says $B(2) = 10{,}000 = B_0 \cdot e^{2k}$ Taking logs yields $\ln 10{,}000 = \ln B_0 + 2k$ ①
$B(5) = 70{,}000 = B_0 e^{5k}$ $\qquad \ln 70{,}000 = \ln B_0 + 5k$ ②
We don't need k but we do need B_0, so we multiply ① by $5/2$ and subtract ② from it:

$$\tfrac{5}{2}\ln 10{,}000 - \ln 70{,}000 = \left(\tfrac{5}{2} - 1\right)\ln B_0 \quad \Rightarrow \quad \ln B_0 = \tfrac{5}{3}\ln 10{,}000 - \tfrac{2}{3}\ln 70{,}000$$

Simplifying the logs gives $\ln B_0 = \ln\left(\dfrac{10000^{5/3}}{70{,}000^{2/3}}\right) \Rightarrow B_0 = \dfrac{10{,}000^{5/3}}{70{,}000^{2/3}} = \dfrac{(10^4)^{5/3}}{7^{2/3}\,(10^4)^{2/3}}$

$\boxed{B_0 = \dfrac{10{,}000}{7^{2/3}}}$ $\quad (7^{2/3} = \sqrt[3]{49})$.

4. (a) (i) $\tan^{-1}x = O(1)$. TRUE. $\tan^{-1}x$ is a function from $\mathbb{R}$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$ and approaches $\frac{\pi}{2}$ as $x \to \infty$. We need to find M such that $\dfrac{\tan^{-1}x}{1} \le M$ for x large enough
$\tan^{-1}x < \frac{\pi}{2} < 2$, so $M = 2$ will do.

(ii) $x^{-2} \cdot 3^x$ grows slower than $x \cdot 2^x$. FALSE. If it were the case, we would need $\lim\limits_{x\to\infty} \dfrac{x^{-2}3^x}{x \cdot 2^x} = 0$.
However $\lim\limits_{x\to\infty} \dfrac{x^{-2}3^x}{x\,2^x} = \lim\limits_{x\to\infty} \dfrac{(3/2)^x}{x^3} = \lim\limits_{x\to\infty} \dfrac{(3/2)^x \ln(3/2)}{3x^2} = \lim\limits_{x\to\infty} \dfrac{(3/2)^x (\ln 3/2)^2}{6x}$

$= \lim\limits_{x\to\infty} \dfrac{(3/2)^x(\ln 3/2)^3}{6} = \infty$ (repeated applications of L'Hospital's rule

(iii) $\log_2 3^{x^2}$ grows at the same rate as $(x+7)^2$. TRUE: $\log_2 3^{x^2} = x^2 \cdot \log_2 3$

$$\lim_{x\to\infty} \frac{(x+7)^2}{\log_2 3^{x^2}} = \lim_{x\to\infty} \frac{(x+7)^2}{x^2 \log_2 3} = \lim_{x\to\infty} \left(1 + \frac{7}{x}\right)^2 \cdot \frac{1}{\log_2 3} = \frac{1}{\log_2 3}$$ finite and nonzero.

(iv) $\frac{1}{x} = o\left(\frac{1}{\ln x}\right)$ TRUE:

$$\lim_{x\to\infty} \frac{1/x}{1/\ln x} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{(L'H)}{=} \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

(b) $f = O(g)$ and $g = O(h)$. Then $f = O(h)$: $f = O(g)$ means there is a positive integer M_1 such that $\frac{f(x)}{g(x)} \leq M$ for x large enough (say, $x \geq N_1$)

Similarly, $g = O(h)$ implies that there is M_2 with $\frac{g(x)}{h(x)} \leq M_2$ for x large enough (say, $x \geq N_2$)

Then for $x > N_1, x > N_2$, $\frac{f(x)}{h(x)} = \frac{f(x)}{g(x)} \cdot \frac{g(x)}{h(x)} \leq M_1 \cdot M_2$ and $f = O(h)$.

5. (a) $I = \int x e^{-x^2} dx$. Set $u = -x^2$. Then $du = -2x\,dx$ ($x = \frac{-du}{2}$). So $I = \int -\frac{1}{2} \cdot e^u du = -\frac{1}{2} e^u + C$

In x: $\boxed{I = -\frac{1}{2} e^{-x^2} + C}$.

(b) $I = \int \frac{dx}{\sqrt{x - x^2}}$. We start by completing the square: $(x - \frac{1}{2})^2 = x^2 - x + \frac{1}{4}$. So $x - x^2 = \frac{1}{4} - (x - \frac{1}{2})^2$

So $I = \int \frac{dx}{\sqrt{\frac{1}{4} - (x - \frac{1}{2})^2}}$. Now set $u = \frac{(x - 1/2)}{1/2} = 2x - 1$, $du = 2dx$. Then $(x - 1/2)^2 = \frac{1}{4} u^2$

$$I = \int \frac{du/2}{\sqrt{\frac{1}{4} - \frac{1}{4}u^2}} = \frac{1/2}{1/2} \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + C. \text{ In } x: \boxed{I = \sin^{-1}(2x - 1) + C}.$$

(c) $I = \int x^a \ln x\, dx$ $(a \neq -1)$. We integrate by parts with $dv = x^a . dx$, $u = \ln x$

Then $v = \frac{1}{a+1} x^{a+1}$ since $a \neq -1$ and $du = \frac{1}{x} dx$.

$$I = \int u\,dv = uv - \int v\,du = \frac{1}{a+1} \cdot x^{a+1} \ln x - \int \frac{1}{a+1} x^{a+1} \cdot \frac{dx}{x} = \frac{1}{a+1} x^{a+1} \ln x - \frac{1}{a+1} \int x^a dx$$

$$\text{So } I = \frac{1}{a+1} x^{a+1} \ln x - \frac{1}{(a+1)^2} \cdot x^{a+1} + C = \boxed{\frac{x^{a+1}}{a+1} \left(\ln x - \frac{1}{a+1} \right) + C}.$$

(d) $I = \int \frac{2x - 1}{x^2 + 2x + 2} dx$. First note that $\frac{d}{dx}(x^2 + 2x + 2) = 2x + 2$. This suggests splitting the numerator as

$2x - 1 = (2x + 2) - 3$. Then note that $x^2 + 2x + 2 = (x^2 + 2x + 1) + 1 = (x + 1)^2 + 1^2$.

$$I = \int \frac{(2x + 2) - 3}{x^2 + 2x + 2} dx = \int \frac{2x - 2}{x^2 + 2x + 2} dx - 3 \int \frac{dx}{(x+1)^2 + 1^2} = \boxed{\ln|x^2 + 2x + 2| - 3\tan^{-1}(x + 1) + C}.$$

6. (a) $I_0 = \int_0^{\pi/4} \tan^0 x\, dx = \int_0^{\pi/4} dx = x \Big]_0^{\pi/4} = \pi/4$. $I_1 = \int_0^{\pi/4} \tan x\, dx = \int_0^{\pi/4} \frac{-\frac{d}{dx}\cos x}{\cos x} dx = -\ln|\cos x| \Big]_0^{\pi/4}$

$= -\ln \cos \pi/4 - (-\ln 1) = -\ln \frac{1}{\sqrt{2}} = \ln \sqrt{2} = \frac{1}{2} \ln 2$

(b) $I_{n+2} = \int_0^{\pi/4} \tan^n x \cdot \tan^2 x\, dx = \int_0^{\pi/4} \tan^n x \cdot (\sec^2 x - 1) = \int_0^{\pi/4} \underbrace{\tan^n x}_{u^n} \underbrace{\sec^2 x\, dx}_{du \text{ if } u = \tan x} - I_n = \frac{\tan^{n+1} x}{n+1} \Big]_0^{\pi/4} - I_n$

$$\boxed{I_{n+2} = \frac{1}{n+1} - I_n}$$

(c) $I_{n+4} = \frac{1}{n+3} - I_{n+2} = \frac{1}{n+3} - \left(\frac{1}{n+1} - I_n\right) = I_n - \frac{1}{n+1} + \frac{1}{n+3} = I_n - \frac{2}{(n+1)(n+3)}$

Therefore $I_4 = I_0 - \frac{2}{(0+1)(0+3)} = \frac{\pi}{4} - \frac{2}{3}$; $I_5 = I_1 - \frac{2}{(1+1)(1+3)} = \frac{1}{2} \ln 2 - \frac{2}{8} = \frac{1}{2} \ln 2 - \frac{1}{4}$