Practice Prelim 3 Solutions, Math 191, Fall 2005

Problem 1) Decide, giving reasons, whether the following series converges absolutely, converges conditionally, or diverges?

Note: all the series have positive terms, so either converge absolutely, or diverge. There is no conditional convergence.

$$a) \qquad \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}} \qquad [10\, points]$$

$\frac{(\ln n)^2}{n^{3/2}} = \frac{1}{n^{\frac{5}{4}}} \cdot \left(\frac{\ln n}{n^{\frac{1}{8}}}\right)^2$ Since $\left(\frac{\ln n}{n^{\frac{1}{8}}}\right) \to 0$ by L'Hopital's rule as $n \to \infty$ we see that for n large enough $\frac{(\ln n)^2}{n^{3/2}} = \frac{1}{n^{\frac{5}{4}}} \cdot \left(\frac{\ln n}{n^{\frac{1}{8}}}\right)^2 \leq \frac{1}{n^{\frac{5}{4}}}$.Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{5}{4}}}$ converges as it is a p-series with $p = \frac{5}{4} > 1$, it follows by the Comparison Test that $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$ also converges.

$$b) \sum_{n=2}^{\infty} \frac{1}{n + \sin n} \qquad [5\, points]$$

$\frac{n+\sin n}{n} = 1 + \frac{\sin n}{n} \to 1$ as $n \to \infty$ since $|\sin n|$ is bounded. Comparison with the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$,which diverges, therefore shows by the Limit Comparison Test that

$\sum_{n=2}^{\infty} \frac{1}{n+\sin n}$ also diverges.

$$c) \sum_{n=2}^{\infty} \frac{n\sqrt{n+1}}{n^3 + 3n + 1} \qquad [5\, points]$$

Let a_n be the n^{th} term of the above series, and let b_n be $\frac{1}{n^{\frac{3}{2}}}$.Then $\frac{a_n}{b_n} = \frac{n^{\frac{5}{2}}\sqrt{n+1}}{n^3+3n+1} = \frac{n^3\sqrt{\frac{n+1}{n}}}{n^3+3n+1}$. Dividing both the numerator and the denominator by n^3 shows that $\frac{a_n}{b_n} = \frac{\sqrt{\frac{n+1}{n}}}{1+\frac{3}{n}+\frac{1}{n^2}}$. ¿From this it follows $\frac{a_n}{b_n} \to 1$ as $n \to \infty$. Since $\sum_{n=2}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges (because it is a p-series with $p = \frac{3}{2} > 1$), it follows from the Limit Comparison Test that $\sum_{n=2}^{\infty} \frac{n\sqrt{n+1}}{n^3+3n+1}$ also converges.

$$d) \sum_{n=1}^{\infty} \frac{(2n+1)!^2}{(3n)!} \qquad [10\, points]$$

Let a_n be the n^{th} term of the above series. Then $\frac{a_{n+1}}{a_n} = \frac{(2n+3)!^2}{(3n+3)!} \cdot \frac{(3n)!}{(2n+1)!^2} = \frac{(2n+2)^2(2n+3)^2.}{(3n+1)(3n+2)(3n+3)}$.The numerator is a 4^{th} degree polynomial in n, while the denominator is of degree 3.So $\frac{a_{n+1}}{a_n} \to \infty$ as $n \to \infty$.By the Ratio Test (or by the n^{th} term test) this shows that the series diverges.

Problem 2 a) The factor $\frac{1}{1+x}$ can be expanded as a geometric series and hence the Maclaurin series for $f(x)$ is:

$$f(x) = x^2\left[1 - x + x^2 - x^3 + ... + (-1)^{n-1}x^{n-1} + ...\right]$$

Hence,

$$f(x) = \left[x^2 - x^3 + x^4 - x^5 + ... + (-1)^{n-1}x^{n+1} + ...\right]$$

By applying the ratio test, it can be seen that the series converges absolutely for $|x| < 1$.

Problem 2 b) At $x = 1$,

$$|f(x)| = 1 + 1 + 1 + ...$$

which diverges. Hence, the series does not converge absolutely at $x = 1$.

Problem 3 a) The series can be split into a sum of two geometric series:

$$1 + \frac{2}{10} + \frac{2}{10^4} + \frac{2}{10^7} + \frac{3}{10^2} + \frac{3}{10^5} + \frac{3}{10^8} + ...$$

Hence, the sum is:

$$Sum = 1 + \frac{2}{10\left[1 - \frac{1}{10^3}\right]} + \frac{3}{10^2\left[1 - \frac{1}{10^3}\right]} \qquad (No \quad need \quad to \quad simplify \quad this)$$

Problem 3 b) This series is in fact an alternating series in disguise! To see this, write out the first few terms...

$$\sum_{n=1}^{\infty}\left(\sin\frac{1}{2n} - \sin\frac{1}{2n+1}\right) = \sin\frac{1}{2} - \sin\frac{1}{3} + \sin\frac{1}{4} - \sin\frac{1}{5} + - + ...$$

Now $\sin\frac{1}{n} > 0$ for all $n \geq 1$, and further,

$$\frac{d}{dx}\sin\frac{1}{x} = -\frac{1}{x^2}\cos\frac{1}{x} < 0 \qquad x > 1.$$

Thus $\left\{\sin\frac{1}{n}\right\}_{n=1}^{\infty}$ is a decreasing nonnegative sequence. Hence the alternating series theorem gives that the series converges.

Problem 4) Evaluate the integral

$$-\int_0^1 \ln x dx.$$

The first thing to notice is that the integrand is singular at $x = 0$, so this is an improper integral. Now we use integration by parts to compute...

$$\begin{aligned}
-\int_0^1 \ln x dx &= -\lim_{a\to 0^+}\int_a^1 \ln x dx \\
&= -\lim_{a\to 0^+}\left(x\ln x|_a^1 - \int_a^1 \frac{x}{x}dx\right) \\
&= -\lim_{a\to 0^+}\left(x\ln x - x|_a^1\right. \\
&= -\lim_{a\to 0^+}\left(1\ln 1 - 1 - a\ln a + a\right) \\
&= 1,
\end{aligned}$$

where in the last statement, we've used the fact that

$$\lim_{a\to 0^+} a\ln a = \lim_{a\to 0^+} \frac{\ln a}{1/a} = \lim_{a\to 0^+} \frac{1/a}{-1/a^2} = 0.$$

Problem 5) According to the error bound formula for Simpson's rule, how many subintervals should you use to be sure of estimating the value of

$$\ln 3 = \int_1^3 \frac{1}{x}dx$$

to an accuracy of better than 10^{-2} in absolute value?

The error formula for Simpson's rule is

$$E_s \le \frac{(b-a)^5}{180n^4} \max_{x\in[a,b]} |f^{(4)}(x)|. \tag{1}$$

Hence we need to compute $f^{(4)}$ for $f(x) = 1/x$:

$$\begin{aligned} f(x) &= \frac{1}{x} \\ f'(x) &= -\frac{1}{x^2} \\ f''(x) &= \frac{2}{x^3} \\ f^{(3)}(x) &= -\frac{6}{x^4} \\ f^{(4)}(x) &= \frac{24}{x^5}. \end{aligned}$$

Now $f^{(4)}$ is evidently a decreasing function of x, so it attains its maximum at the lower endpoint of $[1, 3]$, so

$$\max_{x\in[1,3]} |f^{(4)}(x)| = f^{(4)}(1) = 24.$$

Now, rearranging Equation 1, we have

$$\begin{aligned} n &\ge \left(\frac{(b-a)^5}{180E_s} \max_{x\in[a,b]} |f^{(4)}(x)|\right)^{1/4} \\ &> \left(\frac{2^5}{18010^{-2}}24\right)^{1/4} \approx 4.5449. \end{aligned}$$

Now since the number of subintervals must be even, we require 6 subintervals or more to approximate the integral.